PROPERTIES

OF

EXPANDING UNIVERSES

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Thesis presented for the Degree of Ph.D in the University of Cambridge

ABSTRACT

Some implications and consequences of the expansion of the universe are examined. In Chapter 1 it is shown that this expansion creates grave difficulties for the Hoyle-Narlikar theory of gravitation. Chapter 2 deals with perturbations of an expanding homogeneous and isotropic universe. The conclusion is reached that galaxies cannot be formed as a result of the growth of perturbations that were initially small. The propagation and absorption of gravitational radiation is also investigated in this approximation. In Chapter 3 gravitational radiation in an expanding universe is examined by a method of asymptotic expansions. The 'peeling-off' behaviour and the asymptotic group are derived. Chapter 4 deals with the occurrence of singularities in cosmological models. It is shown that a singularity is inevitable provided that certain very general conditions are satisfied.

INTRODUCTION

The idea that the universe is expanding is of recent origin. All the early cosmologies were essentially stationary and even Einstein whose theory of relativity is the basis for almost all modern developments in cosmology, found it natural to suggest a static model of the universe. However there is a very grave difficulty associated with a static model such as Einstein's which is supposed to have existed for an infinite time. For, if the stars had been radiating energy at their present rates for an infinite time, they would have needed an infinite supply of energy. Further, the flux of radiation now would be infinite. Alternatively, if they had only a limited supply of energy, the whole universe would by now have reached thermal equilibrium which is certainly not the case. This difficulty was noticed by Olbers who however was not able to suggest any solution. The discovery of the recession of the nebulae by Hubble led to the abandonment of static models in favour of ones which were expanding.

Clearly there are several possibilities: the universe may have expanded from a highly dense state a finite time ago (the so-called 'big-bang' model), another is that the present expansion may have been preceded by a contraction which, in its turn may have been preceded by another expansion (the 'bouncing' or oscillating model), however, this model suffers from the same difficulties over entropy as the static model; finally it is possible that the expansion may have been proceeding at much the same rate for an infinite time. It is then necessary to postulate some form of continual creation of matter in order to prevent the expansion from reducing the density to zero. This leads to the 'steady-state' model which although expanding presents the same appearance at all times.

The early cosmologies naturally placed man at or near the centre of the universe, but, since the time of Copernicus we have been demoted to a medium sized planet going round a medium sized star somewhere near the edge of a fairly average galaxy. We are now so humble that we would not claim to occupy any special position. However observations seem to indicate that within experimental error (which is fairly high) galaxies have a spatially isotropic distribution around us. As we are not claiming any special position the distribution must be isotropic about every point. This implies that the distribution must be spatially homogeneous as well as isotropic. Of course this homogeneity and isotropy hold only on a large scale, locally there are considerable departures from both.

Robertson and Walker have shown that the metric of a model that is spatially homogeneous and isotropic may be written in the form:

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$k = 0 \text{ or } \pm 1$$

This form will be used extensively in the following chapters. In Chapter 1 it will be shown that the Hoyle-Narlikar theory of gravitation is incompatible with a metric of this form. In Chapter 2 perturbations of this form will be considered in a linearised approximation and, in Chapter 3 gravitational radiation will be considered in a model which tends asymtotically to this form.

Certain of the Robertson-Walker models possess 'horizons'. There are two types: particle horizons and event horizons. A particle horizon is said to exist when an observer's past light cone does not intersect the world line of every particle in the universe (or extended world line in the case of a particle which has been created). An example of a model with a particle horizon is the Einstein-de Sitter model which has $k = 0 \quad R = t^{\frac{2}{3}}$. This is a 'big-bang' model as indeed are all the Robertson-Walker models that satisfy the Einstein equations:

$$R_{ab} - \tfrac{1}{2} g_{ab} R = T_{ab}$$

and contain matter whose pressure is greater than minus one-third the density. An event horizon exists when there are events that a given observer will never see. The steady-state model ($K = 0$ $R = e^t$) is an example of one with an event horizon. Horizons will be further discussed in Chapter 4 which also deals with the occurrence of singularities of space-time and their connection with topology.

Each chapter is self-contained and has its own references. The following notation is used throughout: space-time is taken to be a Riemannian manifold with metric tensor g_{ij} . This is taken to have signature -2 except in Chapter 2 where, in order to facilitate comparison with previous work, the signature is +2. Covariant differentiation is indicated by a semi-colon. Units are employed in which C, the speed of light, and K, the gravitational constant, equal one.

ACKNOWLEDGEMENTS

I wish to thank my supervisor, Dr. D. W. Sciama for his help and advice during my period of research.

I would also like to thank Mr. B. Carter and Dr. G. F. R. Ellis for many useful discussions. I am indebted to R. G. McLenaghan for the calculation of the Bianchi Identities in Chapter 3.

The research described in this thesis was carried on while I held a Research Studentship from the D.S.I.R.

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15th October 1965 S. W. Hawking

This dissertation is my original work

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CHAPTER 1

The Hoyle-Narlikar Theory of Gravitation

1. Introduction

The success of Maxwell's equations has led to electrodynamics being normally formulated in terms of fields that have degrees of freedom independent of the particles in them. However, Gauss suggested that an action-at-a-distance theory in which the action travelled at a finite velocity might be possible. This idea was developed by Wheeler and Feynman [1,2] who derived their theory from an action-principle that involved only direct interactions between pairs of particles. A feature of this theory was that the 'pseudo'-fields introduced are the half-retarded plus half-advanced fields claculated from the world-lines of the particles. However, Wheeler and Feynman, and, in a different way, Hogarth [3] were able to show that, provided certain cosmological conditions were satisfied, these fields could combine to give the observed field. Hoyle and Narlikar [4] extended the theory to general space-times and obtained similar theories for their 'C'-field [5] and for the gravitational field [6]. It is with these theories that this chapter is concerned.

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It will be shown that in an expanding universe the advanced fields are infinite, and the retarded fields finite. This is because, unlike electric charges, all masses have the same sign.

2. The Boundary Condition

Hoyle and Narlikar derive their theory from the action:

$$A = \sum_{a \neq b}\sum \iint G(a,b)\, da\, db,$$

where the integration is over the world-lines of particles $a, b \ldots$. In this expression G is a Green function that satisfies the wave equation:

$$G(X,X')_{;ij}\, g^{ij} + \frac{1}{6} R\, G(X,X') = \frac{\delta^4(X,X')}{\sqrt{-g}}$$

where g is the determinant of g_{ij}. Since the double sum in the action A is symmetrical between all pairs of particles a, b, only that part of $G(a,b)$ that is symmetrical between a and b will contribute to the action i.e. the action can be written

$$A = \sum_{a \neq b}\sum \iint G^*(a,b)\, da, db$$

where $G^*(a,b) = \frac{1}{2} G(a,b) + \frac{1}{2} G(b,a)$.

Thus G^* must be the time-symmetric Green function, and can be written: $G^* = \frac{1}{2} G_{ret} + \frac{1}{2} G_{adv}$ where G_{ret}

and G_{adv} are the retarded and advanced Green functions. By requiring that the action be stationary under variations of the g_{ij}, Hoyle and Narlikar obtain the field-equations:

$$\Big[\sum_{a \neq b}\sum \tfrac{1}{6} m^{(a)}(X) m^{(b)}(X)\Big](R_{ik} - \tfrac{1}{2} g_{ik} R)$$

$$= -T_{ik} + \sum_{a \neq b}\sum \tfrac{1}{3}\Big[m^{(a)}(g_{ik} m^{(b);r}_{\;\;;r} - m^{(b)}_{;ik}) + 2(m^{(a)}_{;i} m^{(b)}_{;k} - \tfrac{1}{4} g_{ik} m^{(a);r} m^{(b)}_{;r}\Big],$$

where $m^{(a)}(x) = \int G^*(x,a)\,da$. However, as a consequence of the particular choice of Green function, the contraction of the field-equations is satisfied identically. There are thus only 9 equations for the 10 components of g_{ij} and the system is indeterminate.

Hoyle and Narlikar therefore impose $\sum m^{(a)} = m_0$ =const., as the tenth equation. By then making the 'smooth-fluid' approximation, that is by putting $\sum_{a \neq b}\sum m^{(a)} m^{(b)} \simeq m_0^2$, they obtain the Einstein field-equations:

$$\tfrac{1}{6} m_0^2 (R_{ik} - \tfrac{1}{2} R g_{ik}) = -T_{ik}$$

Thereis an important difference, however, between these field-equations in the direct-particle interaction theory and in the usual general theory of relativity. In the general theory of relativity, any metric that satisfies the

the field-equations is admissible, but in the direct-particle interaction theory only those solutions of the field-equations are admissible that satisfy the additional requirement:

$$m_0(x) = \sum m^{(a)}(x) = \sum \int G^*(x,a)\,da$$

$$= \tfrac{1}{2}\sum \int G_{ret.}(x,a)\,da + \tfrac{1}{2}\sum \int G_{adv.}(x,a)\,da$$

This requirement is highly restrictive; it will be shown that it is not satisfied for the cosmological solutions of the Einstein field-equations, and it appears that it cannot be satisfied for any models of the universe that either contain an infinite amount of matter or undergo infinite expansion.

The difficulty is similar to that occurring in Newtonian theory when it is recognized that the universe might be infinite.

The Newtonian potential ϕ obeys the equation:

$$\Box \phi = -\kappa \rho \qquad (\rho > 0),$$

where ρ is the density.

In an infinite static universe, ϕ would be infinite, since the source always has the same sign. The difficulty was resolved when it was realized that the universe was expanding, since in an expanding universe the retarded solution of the above equation is finite by a sort of 'red-shift' effect. The advanced solution will be infinite by a 'blue-shift' effect. This is unimportant in Newtonian theory, since one is free to choose the solution of the equation and so may ignore the infinite advanced solution and take simply the finite retarded solution.

Similarly in the direct-particle interaction theory the m-field satisfies the equation:

$$\Box m + \frac{1}{6} R m = N \qquad (N > 0),$$

where N is the density of world-lines of particles. As in the Newtonian case, one may expect that the effect of the expansion of the universe will be to make the retarded solution finite and the advanced solution infinite. However, one is now not free to choose the finite retarded solution, for the equation is derived from a direct-particle interaction action-principle symmetric between pairs of particles, and one must choose for m half the sum of the retarded and advanced solutions. We would expect this to be infinite, and this is shown to be so in the next section.

3. The Cosmological Solutions

The Robertson-Walker cosmological metrics have the form

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right].$$

Since they are conformally flat, one can choose coordinates in which they become

$$\begin{aligned} ds^2 &= \Omega^2[d\tau^2 - d\rho^2 + \rho^2\{d\theta^2 + \rho^2\sin^2\theta\, d\phi^2\}] \\ &= \Omega^2\eta_{ab}\,dx^a dx^b \end{aligned}$$

where η_{ab} is the flat-space metric tensor and

$$\Omega = \Omega(\tau,\rho) = \frac{R(t)}{\sqrt{\{[1+\frac{1}{4}K(\tau+\rho)^2][1+\frac{1}{4}K(\tau-\rho)^2]\}}} \qquad (7)$$

For example, for the Einstein-de Sitter universe

$$K = 0, \quad R(t) = \left(\frac{t}{T}\right)^{\frac{2}{3}} \quad (0 < t < \infty),$$

$$\Omega = R = \left(\frac{\tau}{T}\right)^2 \qquad (0 < \tau < \infty),$$

$$r = \rho \qquad (\tau = T^{\frac{2}{3}}t^{\frac{1}{3}})$$

For the steady-state (de Sitter) universe

$$K = 0, \quad R(t) = e^{\frac{t}{T}} \quad (-\infty < t < \infty)$$

$$\Omega = R = -\frac{T}{\tau} \quad (-\infty < \tau < 0)$$

$$r = \rho \quad (\tau = -Te^{\frac{-t}{T}})$$

The Green function $G^*(a,b)$ obeys the equation

$$\Box G^*(a,b) + \frac{1}{6} R G^*(a,b) = \frac{\delta^4(a,b)}{\sqrt{-g}}$$

From this it follows that

$$\frac{1}{\Omega^4} \frac{\partial}{\partial x^a}\left(\Omega^2 \eta^{ab} \frac{\partial}{\partial x^b} G^*\right) + \frac{\partial}{\partial x^a}\left(\eta^{ab}\frac{\partial}{\partial x^b}\Omega\right)\Omega^{-3} G^*$$
$$= \Omega^{-4}\delta^4(a,b)$$

If we let $G^* = \Omega^{-1} S$, then

$$\Omega \frac{\partial}{\partial x^a}\left(\eta^{ab}\frac{\partial}{\partial x^b} S\right) = \delta^4(a,b)$$

This is simply the flat-space Green function equation, and hence

$$G^*(\tau_1, 0; \tau_2 \rho) = \frac{\Omega^{-1}(\tau_1)}{8\pi}\left[\frac{\delta(\rho - \tau_2 + \tau_1)}{\Omega(\tau_2)\rho} + \frac{\delta(\rho + \tau_2 - \tau_1)}{\Omega(\tau_2)\rho}\right].$$

The 'm'-field is given by

$$m(\tau_1) = \int G^* N \sqrt{-g}\, dx^4 = \tfrac{1}{2}(m_{ret.} + m_{adv.}).$$

For universes without creation (e.g. the Einstein-de Sitter universe), $N = R^{-3} n$; $n =$ const. For universes with creation (steady state) $N = n$, $n =$ const.

$$m_{adv.}(\tau_1) = \Omega^{-1}(\tau_1)\int \frac{N\Omega^3(\tau_2)}{4\pi r} 4\pi r^2 dr$$

where the integration is over the future light cone. This will normally be infinite in an expanding universe, e.g. in the Einstein-de Sitter universe.

$$m_{adv}(\tau_1) = \left(\frac{\tau_1}{T}\right)^{-2}\int_{\tau_1}^{\infty} n(\tau_2 - \tau_1)\, d\tau_2$$

$$= \infty$$

In the steady-state universe

$$m_{adv.}(\tau_1) = \left(\frac{-T}{\tau_1}\right)^{-1}\int_{\tau_1}^{0} - n\left(\frac{T}{\tau_2}\right)^3 (\tau_2 - \tau_1)\, d\tau_2.$$

$$= \infty$$

By contrast, on the other hand, we have

$$m_{ret}(\tau_1) = \Omega^{-1}(\tau_1)\int \frac{N\Omega^3}{4\pi r} 4\pi r^2 dr$$

where the integration is over the past light cone. This will normally be finite, e.g. in the Einstein-de Sitter universe

$$m_{ret.}(\tau_1) = \left(\frac{\tau_1}{T}\right)^{-2} \int_0^{\tau_1} -n(\tau_2 - \tau_1)\, d\tau_2 = \tfrac{1}{2} n T^2,$$

while in the steady-state universe

$$m_{ret}(\tau_1) = \left(\frac{-T}{\tau_1}\right)^{-1} \int_{-\infty}^{\tau_1} n \left(\frac{T}{\tau_2}\right)^3 (\tau_2 - \tau_1)\, d\tau_2$$
$$= \tfrac{1}{2} n T^2$$

Thus it can be seen that the solution $m =$ const. of the equation

$$\Box m + \tfrac{1}{6} R m = N$$

is not, in a cosmological metric, the half-advanced plus half-retarded solution since this would be infinite. In fact, in the case of the Einstein-de Sitter and steady-state metrics, it is the pure retarded solution.

4. The 'C'-Field

Hoyle and Narlikar derive their direct-particle interaction theory of the 'C'-field from the action

$$A = \sum_{a \neq b}\sum \iint \hat{G}(a,b)_{;i_a k_b}\, da^i\, db^k,$$

where the suffixes a, b refer to differentiation of $\hat{G}(a,b)$ on the world-lines of a, b respectively. $\hat{G}$ is a Green function obeying the equation

$$\Box \hat{G}(X, X') = \frac{\delta^4(X, X')}{\sqrt{-g}}.$$

We define the 'C'-field by

$$C(x) = \sum \int \hat{G}(x, a)_{;i_a} \, da^i,$$

and the matter-current J^k by

$$J^k(y) = \sum \int \delta^4(y, b) \, db^k.$$

Then

$$C(x) = \int \hat{G}(x, y) J^k(y)_{;k} \sqrt{-g} \, dx^4,$$

$$\Box C = J^k{}_{;k}$$

We thus see that the sources of the 'C'-field are the places where matter is created or destroyed.

As in the case of the 'm'-field, the Green function must be time-symmetric, that is

$$\hat{G}(a, b) = \tfrac{1}{2} \hat{G}_{ret.}(a, b) + \tfrac{1}{2} \hat{G}_{adv}(a, b)$$

Hoyle and Narlikar claim that if the action of the 'C'-field is included along with the action of the 'm'-field, a universe will be obtained that approximates to the steady-state universe on a large scale although there may be local irregularities. In this universe, the value of C will be finite and its gradient time-like and of unit magnitude.

Given this universe, we may check it for consistency by claculating the advanced and retarded 'C'-fields and finding if their sum is finite. We shall not do this directly but will show that the advanced field is infinite while the retarded field is finite.

Consider a region in space-time bounded by a three-dimensional space-like hypersurface D at the present time, and the past light cone Σ of some point P to the future of D.

By Gauss's theorem

$$\int_V \Box C \sqrt{-g}\, dx^4 = \int_{\Sigma + D} \frac{\partial C}{\partial n}\, dS$$

$$= \int J^K{}_{;K} \sqrt{-g}\, dx^4$$

Let the advanced field produced by sources within V be C'. Then C' and $\frac{\partial C'}{\partial n}$ will be zero on Σ, and hence

$$\int_V J^k{}_{;k} \sqrt{-g}\, dx^4 = \int_D \frac{\partial C'}{\partial n} dS .$$

But $J^k{}_{;k}$ is the rate of creation of matter= n (const.) in the steady-state universe, and hence

$$\int_D \frac{\partial C'}{\partial n} dS = nV .$$

As the point P is taken further into the future, the volume of the region V tends to infinity. However, the area of the hypersurface D tends to a finite limit owing to horizon effects. Therefore the gradient $\frac{\partial C'}{\partial n}$ must be infinite. A similar calculation shows the gradient of the retarded field to be finite. Their sums cannot therefore give the field of unit gradient required by the Hoyle-Narlikar theory.

It is worth noting that this result was obtained without assumptions of a smooth distribution of matter or of conformal flatness.

5. Conclusion

It is one of the weaknesses of the Einstein theory of relativity that although it furnishes field equations it does not provide boundary conditions for them. Thus it does not give a unique model for the universe but allows a whole series of models. Clearly a theory that provided boundary conditions and thus restricted the possible solutions would be very attractive. The Hoyle-Narlikar theory does just that(the requirement that $m = \frac{1}{2} m_{ret} + \frac{1}{2} m_{adv}$ is equivalent to a boundary condition). Unfortunately, as we have seen above, this condition excludes those models that seem to correspond to the actual universe, namely the Robertson-Walker models.

The calculations given above have considered the universe as being filled with a uniform distribution of matter. This is legitimate if we are able to make the 'smooth-fluid' approximation to obtain the Einstein equations. Alternatively if this approximation is invalid, it cannot be said that the theory yields the Einstein equations.

It might possibly be that local irregularities could make m_{adv} finite, but this has certainly not been demonstrated and seems unlikely in view of the fact that, in the Hoyle-Narlikar direct-particle interaction theory of their 'C'-field, which is derived from a very similar action-principle, it can be shown without assuming a smooth distribution that the advanced 'C' field will be infinite in an expanding universe with creation.

The reason that it is possible to formulate a direct-particle interaction theory of electrodynamics that does not encounter this difficulty of having the advanced solution infinite is that in electrodynamics there are equal numbers of sources of positive and negative sign. Their fields can cancel each other out and the total field can be zero apart from local irregularities. This suggest that a possible way to save the Hoyle-Narlikar theory would be to allow masses of both positive and negative sign. The action would be

$$A = \sum_{a \neq b}\sum q_a q_b \iint G^*(a,b)\, da\, db \quad (q_a, q_b = \pm 1$$

where q_a, q_b are gravitational charges analogous to electric charges. Particles of positive q in a positive 'm'-field and particles of negative q in a negative 'm'-field would have the normal gravitational properties, that is, they would have positive gravitational and inertial masses.

A particle of negative q in a positive 'm'-field would still follow a geodesic. Therefore it would be attracted by a particle of positive q. Its own gravitational effect however would be to repel all other particles. Thus it would have the properties of the negative mass described by Bondi[8] that is, negative gravitational mass and negative inertial mass.

Since there does not seem to be any matter having these properties in our region of space (where $m \simeq$ const. > 0) there must clearly be separation on a very large scale. It would not be possible to identify particles of negative q with antimatter, since it is known that antimatter has positive inertial mass. Hwever, the introduction of negative masses would probably raise more difficulties than it would solve.

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CHAPTER 2

PERTURBATIONS

1. Introduction

Perturbations of a spatially isotropic and homogeneous expanding universe have been investigated in a Newtonian approximation by Bonnor[1] and relativistically by Lifshitz[2], Liftshitz and Khalatnikov[3] and Irvine[4]. Their method was to consider small variations of the metric tensor. This has the disadvantage that the metric tensor is not a physically significant quantity, since one cannot directly measure it, but only its second derivatives. It is thus not obvious what the physical interpretation of a given perturbation of the metric is. Indeed it need have no physical significance at all, but merely correspond to a coordinate transformation. Instead it seems preferable to deal in terms of perturbations of the physically significant quantity, the curvature.

2. Notation

Space-time is represented as a four-dimensional Riemannian space with metric tensor g_{ab} of signature +2. Covariant differentiation in this space is indicated by a semi-colon. Square brackets around indices indicate antisymmetrisation and round brackets symmetrisation. The conventions for the Riemann and Ricci tensors are:-

$$V_{a;[bc]} = 2R^{p}{}_{acb}V_{p} \, ,$$

$$R_{ab} = R_{a}{}^{p}{}_{bp}$$

η_{abcd} is the alternating tensor.

Units are such that k the gravitational constant and c, the speed of light are one.

3. The Field Equations

We assume the Einstein equations:

$$R_{ab} - \frac{1}{2} g_{ab} R = - T_{ab}$$

where T_{ab} is the energy momentum tensor of matter. We will assume that the matter consists of a perfect fluid. Then,

$$T_{ab} = \mu U_a U_b + p h_{ab}$$

where U_a is the velocity of the fluid, $U_a U^a = -1$:

μ is the density

p is the pressure

$h_{ab} = g_{ab} + U_a U_b$ is the projection operator into the hyperplane orthogonal to U_a :

$$h_{ab} U^b = 0.$$

We decompose the gradient of the velocity vector U_a as

$$U_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3} h_{ab} \theta - \dot{U}_a U_b$$

where $\dot{U}_a = U_{a;b} U^b$ is the acceleration,

$\theta = U_a{}^{;a}$ is the expansion,

$\sigma_{ab} = U_{(c;d)} h^c_a h^d_b - \frac{1}{3} h_{ab} \theta$ is the shear,

$\omega_{ab} = U_{[c;d]} h^c_a h^d_b$ is the rotation of the flow lines U_a. We define the rotation vector ω_a as

$$\omega_a = \frac{1}{2} \eta_{abcd} \omega^{cd} U^b .$$

We may decompose the Riemann tensor R_{abcd} into the Ricci tensor R_{ab} and the Weyl tensor C_{abcd} :

$$R_{abcd} = C_{abcd} - g_{a[d} R_{c]b} - g_{b[c} R_{d]a} - R/3\, g_{a[c} g_{d]b} ,$$

$$C_{abcd} = C_{[ab][cd]} ,$$

$$C^a{}_{\cdot bca} = 0 = C_{a[bcd]} .$$

C_{abcd} is that part of the curvature that is not determined locally by the matter. It may thus be taken as representing the free gravitational field (Jordan, Ehlers and Kundt[5]). We may decompose it into its "electric" and "magnetic" components.

$$E_{ab} = - C_{abpq} u^p u^q ,$$

$$H_{ab} = -\tfrac{1}{2} C_a{}^{pqr} \eta_{qrbs} u_p u^s ,$$

$$C_{ab}{}^{cd} = 8 u_{[a} E_{b]}{}^{[c} u^{d]} - 4 \delta^{[c}_{[a} E^{d]}_{b]} ,$$

$$- 2 \eta_{abcd} u^p H^{q[c} u^{d]} - 2 \eta^{cdrs} u_r H_{s[a} u_{b]} ,$$

$$E_{ab} = E_{(ab)} , \qquad H_{ab} = H_{(ab)} ,$$

$$E_a{}^a = H_a{}^a = 0$$

$$E_{ab} u^b = H_{ab} u^b = 0 ,$$

E_{ab} and H_{ab} each have five independent components.
We regard the Bianchi identities,

$$R_{ab[cd;e]} = 0$$

as field equations for the free gravitational field.
Then

$$C_{abcd}{}^{;d} = - R_{c[b;a]} + \tfrac{1}{6} g_{c[b} R_{;a]}$$ (Kundt and Trümper,[6]).

Using the decompositions given above, we may write these in a form analogous to the Maxwell equations.

$$h_a{}^b E_{bc;d} h^{cd} + 3 H_{ab}\omega^b - \eta_{abcd} u^b \sigma^c{}_e H^{de} = \tfrac{1}{3} h_a{}^b \mu_{;b} \quad , \tag{1}$$

$$h_a{}^b H_{bc;d} h^{cd} - 3 E_{ab}\omega^b - \eta_{abcd} u^b \sigma^c{}_e E^{de} = (\mu + p)\,\omega_a \quad , \tag{2}$$

$$\begin{aligned} \perp \dot{E}_{ab} &+ h_{(a}{}^f \eta_{b)cde} u^c H_f{}^{d;e} + E_{ab}\theta - E^c{}_{(a}\omega_{b)c} \\ &- E^c{}_{(a}\sigma_{b)c} - \eta_{acde}\,\eta_{bpqr} u^c u^p \sigma^{dq} E^{er} \\ &+ 2 H^d{}_a \eta_{bcde} u^c \dot{u}^e = -\tfrac{1}{2}(\mu + p)\,\sigma_{ab} \quad , \end{aligned} \tag{3}$$

$$\begin{aligned} \perp \dot{H}_{ab} &- h_{(a}{}^f \eta_{b)cde} u^c E_f{}^{d;e} + H_{ab}\theta - H^c{}_{(a}\omega_{b)c} \\ &- H^c{}_{(a}\sigma_{b)c} - \eta_{acde}\,\eta_{bpqr} u^c u^p \sigma^{dq} H^{er} \\ &+ 2 H^d{}_a \eta_{bcde} u^c \dot{u}^e = 0 \quad . \end{aligned} \tag{4}$$

where $\perp$ indicates projection by h_{ab} orthogonal to U_a. (c.f. Trümper,[7]).

The contracted Bianchi identities give,

$$\left(R_{ab} - \tfrac{1}{2} g_{ab} R\right)^{;b} = -T_{ab}{}^{;b} = 0 \quad ,$$

$$\dot{\mu} + (\mu + p)\,\theta = 0 \quad , \tag{5}$$

$$(\mu + p)\,\dot{u}_a + p_{;b}\, h^b{}_a = 0 \quad . \tag{6}$$

The definition of the Riemann tensor is,

$$u_{a;[bc]} = 2 R_{apbc} u^p \quad .$$

Using the decompositions as above we may obtain what may be regarded as "equations of motion".

$$\dot{\theta} = 2\omega^2 - 2\sigma^2 - \tfrac{1}{3}\theta^2 + \dot{u}_a{}^{;a} - \tfrac{1}{2}(\mu + 3p) \quad , \tag{7}$$

$$\perp \dot{\omega}_{ab} = -\tfrac{2}{3}\omega_{ab}\theta + 2\sigma_{c[a}\omega_{b]}{}^{c} + \dot{u}_{[p;q]} h^p_a h^q_b \quad , \tag{8}$$

$$\begin{aligned} \perp \dot{\sigma}_{ab} = {} & E_{ab} - \omega_{ac}\omega^c{}_b - \sigma_{ac}\sigma^c{}_b - \tfrac{2}{3}\sigma_{ab}\theta \\ & - \tfrac{1}{3} h_{ab}(2\omega^2 - 2\sigma^2 + \dot{u}_c{}^{;c}) + \dot{u}_a \dot{u}_b \\ & + \dot{u}_{(p;q)} h^p_a h^q_b \quad , \end{aligned} \tag{9}$$

where $\quad 2\omega^2 = \omega_{ab}\omega^{ab} \quad , \quad 2\sigma^2 = \sigma_{ab}\sigma^{ab}$

We also obtain what may be regarded as equations of constraint.

$$\theta_{;b} h^b_a = \tfrac{3}{2}\left[(\omega^b{}_{c;b} + \sigma^b{}_{c;b}) h^c_a - \dot{u}^b(\omega_{ab} + \sigma_{ab})\right] \quad , \tag{10}$$

$$\omega_a{}^{;a} = 2\omega_a \dot{u}^a \quad , \tag{11}$$

$$H_{ab} = -h^f{}_{(a}\eta_{b)cde} u^c \left[\omega_f{}^{d;e} + \sigma_f{}^{d;e}\right] \quad . \tag{12}$$

We consider perturbations of a universe that in the undisturbed state in conformally flat, that is

$$C_{abcd} = 0 \quad .$$

By equations (1) - (3), this implies,

$$\sigma_{ab} = \omega_{ab} = 0$$
$$h_a{}^b \mu_{;b} = 0 = \theta_{;b} h^b_a \quad .$$

If we assume an equation of state of the form, $p = p(\mu)$, then by (6), (10), $p_{;b} h^b{}_a = 0 = \dot{u}_a$.

This implies that the universe is spatially homogeneous and isotropic since there is no direction defined in the 3-space orthogonal to U_a.

In this universe we consider small perturbations of the motion of the fluid and of the Weyl tensor. We neglect products of small quantities and perform derivatives with respect to the undisturbed metric. Since all the quantities we are interested in with the exception of the scalars, μ, p, θ have unperturbed value zero, we avoid perturbations that merely represent coordinate transformation and have no physical significance.

To the first order the equations (1) - (4) and (7) - (9) are

$$E_{ab}{}^{;b} = \tfrac{1}{3} h_a{}^b \mu_{;b} \quad , \tag{13}$$

$$H_{ab}{}^{;b} = (\mu + p)\, \omega_a \quad , \tag{14}$$

$$\dot{E}_{ab} + E_{ab}\theta + h^f{}_{(a}\eta_{b)cde}\, u^c H_f{}^{d;e} = -\tfrac{1}{2}(\mu + p)\, \sigma_{ab} \quad , \tag{15}$$

$$\dot{H}_{ab} + H_{ab}\theta - h^f{}_{(a}\eta_{b)cde}\, u^c E_f{}^{d;e} = 0 \quad , \tag{16}$$

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 + \dot{u}_a{}^{;a} - \tfrac{1}{2}(\mu + 3p) \quad , \tag{17}$$

$$\dot{\omega}_{ab} = -\tfrac{2}{3}\omega_{ab}\theta + \dot{u}_{[p;q]}\, h^p_a h^q_b \quad , \tag{18}$$

$$\dot{\sigma}_{ab} = E_{ab} - \tfrac{2}{3}\sigma_{ab}\theta - \tfrac{1}{3} h_{ab}\dot{u}_c{}^{;c} + \dot{u}_{(p;q)}\, h^p_a h^q_b \quad . \tag{19}$$

From these we see that perturbations of rotation or of E_{ab} or H_{ab} do not produce perturbations of the expansion or the density. Nor do perturbations of E_{ab} and H_{ab} produce rotational perturbations.

4. The Undisturbed Metric

Since in the unperturbed state the rotation and acceleration are zero, U_a must be hypersurface orthogonal.

$$u_a = \tau_{;a} \quad ,$$

where τ measures the proper time along the world lines. As the surfaces τ = constant are homogeneous and isotropic they must be 3-surfaces of constant curvature. Therefore the metric can be written,

$$ds^2 = -d\tau^2 + \Omega^2 d\gamma^2$$

where $\qquad \Omega = \Omega(\tau) \quad ,$

$d\gamma^2$ is the line element of a space of zero or unit positive or negative curvature.

We define t by,

$$\frac{dt}{d\tau} = \frac{1}{\Omega} \quad ,$$

then

$$ds^2 = \Omega^2(-dt^2 + d\gamma^2) \quad .$$

In this metric,

$$u_a = (-\Omega, 0, 0, 0) \quad ,$$

$$\therefore \quad \theta = 3\frac{\dot{\Omega}}{\Omega} = \frac{3\Omega'}{\Omega^2}$$

(prime denotes differentiation with respect to t)

Then, by (5), (7)

$$\dot{\mu} = -(\mu + p)\,\frac{3\dot{\Omega}}{\Omega} \quad , \tag{20}$$

$$3\,\frac{\ddot{\Omega}}{\Omega} = -\frac{1}{2}(\mu + 3p) \quad . \tag{21}$$

If we know the relation between μ and p, we may determine Ω . We will consider the two extreme cases, $p = 0$ (dust) and $p = \frac{\mu}{3}$ (radiation). Any physical situation should lie between these.

For $p = 0$

By (20), $\mu = \frac{M}{\Omega^3}$ $\quad M = \text{const.}$

$$\therefore \quad \frac{3}{M}\,\frac{\ddot{\Omega}}{\Omega} - \frac{1}{2\Omega^3} = 0 \quad ,$$

$$\therefore \quad \frac{3}{M}\,\dot{\Omega}^2 - \frac{1}{\Omega} = E \quad , \qquad E = \text{const.}$$

(a) For E 0 ,

$$\Omega = \frac{1}{2E}\left(\cosh\sqrt{\frac{EM}{3}}\,t - 1\right) , \qquad \tau = \frac{1}{2E}\left(\sqrt{\frac{3}{EM}}\,\sinh\sqrt{\frac{EM}{3}}\,t - t\right);$$

(b) For E = 0,

$$\Omega = \frac{M}{12}\,t^2 \quad , \qquad \tau = \frac{M}{36}\,t^3 \; ;$$

(c) For E 0,

$$\Omega = \frac{-1}{2E}\left(1 - \cos\sqrt{\frac{-EM}{3}}\,t\right) , \qquad \tau = \frac{-1}{2E}\left(t - \sqrt{\frac{3}{-EM}}\,\sin\sqrt{\frac{-EM}{3}}\,t\right) .$$

E represents the energy (kinetic + potential) per unit mass. If it is non-negative the universe will expand indefinitely, otherwise it will eventually contract again.

By the Gauss Codazzi equations *R, the curvature of the hypersurface τ = const., is

$$^{*}R = 2\left(-\tfrac{1}{3}\theta^{2}+\mu\right)$$
$$= -\frac{2EM}{\Omega^{2}}$$

If $E>0$, $\quad {}^{*}R = -\frac{6}{\Omega^{2}}$, $\quad M = \frac{3}{E}$;

$E=0$, $\quad {}^{*}R = 0$;

$E<0$, $\quad {}^{*}R = \frac{6}{\Omega^{2}}$, $\quad M = \frac{-3}{E}$.

For $p = \mu/3$

$$\dot{\mu} = -4\frac{\dot{\Omega}}{\Omega},$$
$$\frac{3\ddot{\Omega}}{\Omega} = -\mu,$$
$$\mu = \frac{M}{\Omega^{4}}, \qquad \therefore \frac{3\dot{\Omega}^{2}}{M} - \frac{1}{\Omega^{2}} = E$$

(a) For $E>0$,

$$\Omega = \frac{1}{E}\sinh t, \quad \tau = \frac{1}{E}(\cosh t - 1), \quad {}^{*}R = -\frac{6}{\Omega^{2}};$$

(b) For $E = 0$,

$$\Omega = t, \quad \tau = \frac{1}{2}t^{2}, \quad {}^{*}R = 0;$$

(c) For E   0,

$$\Omega = -\frac{1}{E}\sin t, \quad \tau = \frac{1}{E}(\cos t - 1), \quad {}^{*}R = \frac{6}{\Omega^{2}}.$$

5. Rotational Perturbations

By (6)
$$\dot{u}_{[c;d]}h^{c}_{a}h^{d}_{b} = -\frac{\omega_{ab}\dot{p}}{\mu+p}$$

$$\therefore \quad \dot{\omega}_{ab} = -\omega_{ab}\left(\tfrac{2}{3}\theta + \frac{\dot{p}}{\mu+p}\right) \quad .$$

For $p = 0$,

$$\omega = \frac{\omega_0}{\Omega^2}$$

For $p = \frac{\mu}{3}$,

$$\dot{\omega} = -\omega\left(\tfrac{2}{3}\theta + \tfrac{1}{4}\frac{\dot{\mu}}{\mu}\right),$$

$$= -\tfrac{1}{3}\omega\theta \quad ,$$

$$\therefore \quad \omega = \frac{\omega_0}{\Omega}$$

Thus rotation dies away as the universe expands. This is in fact a statement of the conservation of angular momentum in an expanding universe.

6. Perturbations of Density

For $p = 0$ we have the equations,

$$\dot{\mu} = -\mu\theta$$

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 - \tfrac{1}{2}\mu$$

These involve no spatial derivatives. Thus the behaviour of one region is unaffected by the behaviour of another. Perturbations will consist in some regions having slightly higher or lower values of E than the average. If the universe as whole has a value of E greater than zero, a small perturbation will still have E greater than zero and will continue to expand. It will not contract to form a galaxy. If the universe has a value of E less than zero, a small perturbation can contract. However it will only begin contracting at a time $\delta\tau$ earlier than the whole universe begins contracting, where

$$\frac{\delta\tau}{\tau_0} = \frac{\delta E}{E_0}$$

τ_0 is the time at which the whole universe begins contracting. There is only any real instability when E = 0. This case is of measure zero relative to all the possible values E can have. However this cannot really be used as an arguement to dismiss it as there might be some reason why the universe should have E = 0. For a region with energy $-\delta E$, in a universe with E = 0

$$\Omega = \frac{1}{4\delta E}\left(t^2 - \frac{t^4}{12} + \cdots\right)$$

$$\tau = \frac{1}{12\delta E}\left(t^3 - \frac{t^5}{20} + \cdots\right)$$

$$\mu = \frac{3}{\delta E\,\Omega^3} = \frac{4}{3}\tau^{-2}\left(1 + \frac{(\delta E)^{2/3}}{2\sqrt{3}}\tau^{2/3} + \cdots\right)$$

For E = 0, $\mu = \frac{4}{3}\tau^{-2}$

Thus the perturbation grows only as $\tau^{2/3}$. This is not fast enough to produce galaxies from statistical fluctuations even if these could occur. However, since an evolutionary universe has a particle horizon (Rindler(8), Penrose(9)) different parts do not communicate in the early stages. This makes it even more difficult for statistical fluctuations to occur over a region until light had time to cross the region.

For $p = \mu/3$

$$\dot{\mu} = -\tfrac{4}{3}\mu\theta$$

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 - \mu + \dot{u}_a{}^{;a}$$

$$\dot{u}_a = -\frac{h^b{}_a \mu_{;b}}{4\mu}$$

As before, a perturbation cannot contract unless it has a negative value of E. The action of the pressure forces make it still more difficult for it to contract. Eliminating θ,

$$\mu\ddot{\mu} - \tfrac{5}{4}\dot{\mu}^2 - \tfrac{4}{3}\mu^3 + \tfrac{4}{3}\mu^2 \dot{u}_a{}^{;a} = 0$$

$$\dot{u}_a{}^{;a} = \dot{u}_{a;b} h^{ab} + \dot{u}_a \dot{u}^a$$

$$= -\frac{1}{4}\,\frac{h^{ac}(h^b{}_a \mu_{;b})_{;c}}{\mu}$$

to our approximation.

$h^{ac}\nabla_c h^b_a \nabla_b$ is the Laplacian in the hypersurface τ = constant. We represent the perturbation as a sum of eigenfunctions $S^{(n)}$ of this operator, where,

$$S^{(n)}{}_{;c} u^c = 0$$

$$h^{ac}(h^b{}_a S^{(n)}{}_{;b})_{;c} = -\frac{n^2}{\Omega^2} S^{(n)}$$

These eigenfunctions will be hyperspherical and pseudohyperspherical harmonics in cases (c) and (a) respectively and plane waves in case (b). In case (c) n will take only discrete values but in (a) and (b) it will take all positive values.

$$\mu = \mu_0\left[\, 1 + \sum_n B^{(n)} S^{(n)} \right]$$

where μ_0 is the undisturbed density.

$$\therefore \quad \ddot{B}^{(n)}\mu_0 - \tfrac{1}{2}\dot{B}^{(n)}\dot{\mu}_0 - B^{(n)}\left[\tfrac{4}{3}\mu_0^2 - \frac{n^2}{3\Omega^2}\mu_0\right] = 0$$

As long as $\mu_0 > \frac{n^2}{4\Omega^2}$, $B^{(n)}$ will grow.

For $\mu_0 \gg \frac{n^2}{4\Omega^2}$

$$B^{(n)} \simeq C\tau + D\tau^{-1}$$

These perturbations grow for as long as light has not had time to travel a significant distance compared to the scale of the perturbation ($\sim \frac{\Omega}{n}$). Until that time pressure forces cannot act to even out perturbations.

When $\frac{n^2}{\Omega} \gg \mu_0$, $\quad B''^{(n)} + B'^{(n)}\frac{\Omega'}{\Omega} + \frac{n^2}{3}B^{(n)} = 0,$

$$\therefore \quad B^{(n)} \simeq C\,\Omega^{-\frac{1}{2}}\, e^{i\frac{n}{\sqrt{3}}t}$$

We obtain sound waves whose amplitude decreases with time. These results confirm those obtained by Lifshitz and Khalatnikov(3).

From the forgoing we see that galaxies cannot form as the result of the growth of small perturbations. We may expect that other non-gravitational forces will have an effect smaller than pressure equal to one third of the density and so will not cause relative perturbations to grow faster than τ. To account for galaxies in an evolutionary universe we must assume there were finite, non-statistical, initial inhomogeneities.

7. The Steady-state Universe

To obtain the steady-state universe we must add extra terms to the energy-momentum tensor. Hoyle and Narlikar[10] use,

$$T_{ab} = \mu u_a u_b + p h_{ab} - C_a C_b + \tfrac{1}{2} g_{ab} C_d C^d . \tag{20}$$

where,

$$C_a = C_{;a} ,$$

$$C_{;a}{}^{a} = - j_a{}^{;a} ,$$

$$j_a = (\mu + p) u_a$$

Since $T_{ab}{}^{;b} = 0$,

$$\dot{\mu} + (\mu + p)\theta + u^a C_a C_b{}^{;b} = 0 \tag{21}$$

$$(\mu + p)\dot{u}_a + p_{;b} h^b{}_a - h^b{}_a C_b C_d{}^{;d} = 0 . \tag{22}$$

There is a difficulty here, if we require that the "C" field should not produce acceleration or, in other words, that the matter created should have the same velocity as the matter already in existence. We must then have

$$h^b{}_a C_b = 0 . \tag{23}$$

However since C is a scalar, this implies that the rotation of the medium is zero. On the other hand if (23) does not hold, the equations are indeterminate (c.f. Raychaudhuri and Bannerjee[11]). In order to have a determinate set of equations we will adopt (23) but drop the requirement that C_a is the gradient of a scalar. The condition (23) is not very satisfactory but it is difficult to think of one more satisfactory. Hoyle and Narlikar[12] seek to avoid this difficulty by taking a particle rather than a fluid picture. However this has a serious drawback since it leads to infinite fields (Hawking[13]).

From (17),

$$C_a = -u_a \left[1 - \frac{\dot{p}}{\dot{\mu} + \dot{p} + (\mu + p)\theta} \right]$$

$$\therefore \quad C_a{}^{;a} = -(\dot{\mu} + \dot{p}) - (\mu + p)\theta \quad ,$$

$$= -\theta \left[1 - \frac{\dot{p}}{\dot{\mu} + \dot{p} + (\mu + p)\theta} \right] + \left(\frac{\dot{p}}{\dot{\mu} + \dot{p} + (\mu + p)\theta} \right)^{\cdot}$$

For $\mu \gg p$

$$(\dot{\mu} + \dot{p}) = \theta\left[1 - (\mu + p)\right]$$

$$\therefore (\mu + p) \rightarrow 1$$

Thus, small perturbations of density die away. Moreover equation (18) still holds, and therefore rotational perturbations also die away. Equation (19) now becomes

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 - \tfrac{1}{2}(\mu + 3p) + 1$$

$$\therefore \theta \rightarrow \sqrt{3(\tfrac{1}{2} - p)}$$

These results confirm those obtained by Hoyle and Narlikar(14). We see therefore that galaxies cannot be formed in the steady-state universe by the growth of small perturbations. However this does not exclude the possibility that there might by a self-perpetuating system of finite perturbations which could produce galaxies. (Sciama(15), Roxburgh and Saffman(16)).

8. Gravitational Waves

We now consider perturbations of the Weyl tensor that do not arise from rotational or density perturbations, that is,

$$E_{ab}{}^{;b} = H_{ab}{}^{;b} = 0$$

Multiplying (15) by $u^c \nabla_c$, and (16) by $h^{a}{}_{(c}\,\eta_{q)}{}^{rbs} u_r \nabla_s$:

we obtain, after a lot of reduction,

$$\ddot{E}_{ab} - \left(E_{cd;e}h^c_f h^d_g h^e_k\right)_{;i} h^{ki} h^f_a h^g_b + \frac{7}{3}\dot{E}_{ab}\theta$$

$$+ E_{ab}\left(\dot{\theta} + \frac{4}{3}\theta^2 + \frac{1}{3}(\mu - 3p)\right) + \sigma_{ab}\left[\frac{1}{3}\theta(\mu + p) + \frac{1}{2}(\dot{\mu} + \dot{p})\right] = 0 \qquad (24)$$

In empty space with a non-expanding congruence U^a this reduces to the usual form of the linearised theory,

$$\Box^2 E_{ab} = 0$$

The second term in (24) is the Laplacian in the hypersurface τ = constant, acting on E_{ab}. We will write E_{ab} as a sum of eigensfunctions of this operator.

$$E_{ab} = \sum A^{(n)} V_{ab}^{(n)}$$

where $\dot{V}_{ab}^{(n)} = 0$,

$$\left(V_{cd;e}h^c_f h^d_g h^e_k\right)_{;i} h^{ki} h^f_a h^g_b = \frac{-n^2}{\Omega^2} V_{ab}^{(n)} ,$$

$$V_{ab}{}^{;b} = 0 , \qquad V_a{}^a = 0 .$$

Then $\dot{E}_{ab} = \sum \frac{A'^{(n)}}{\Omega} V_{ab}^{(n)}$

$$\ddot{E}_{ab} = \sum\left[\frac{A''^{(n)}}{\Omega^2} - \frac{A'^{(n)}\Omega'}{\Omega^3}\right] V_a^{(n)}$$

Similarly, $$\sigma_{ab} = \sum D^{(n)} V_{ab}^{(n)}$$

Then by (19) $$D'^{(n)} = \Omega A^{(n)} - 2 D^{(n)} \frac{\Omega'}{\Omega}$$

Substituting in (24)

$$A''^{(n)} + \frac{6\Omega'}{\Omega} A'^{(n)} + A^{(n)} \left[n^2 + 3\frac{\Omega''}{\Omega} + 6\frac{\Omega'^2}{\Omega^2} + \tfrac{1}{3}(\mu + 3p)\Omega^2 \right]$$
$$+ D^{(n)} \left[\Omega'(\mu + p) + \tfrac{1}{2}\Omega^2(\dot{\mu} + \dot{p}) \right] = 0$$

We may differentiate again and substitute for D',

For $n \gg 1$

and $\Omega \gg \frac{1}{n^2}$

$$A^{(n)} \doteq \frac{1}{\Omega^3} e^{int}$$

so the gravitational field E_{ab} decreases as Ω^{-1} and the "energy" $\frac{1}{2}(E_{ab} E^{ab} + H_{ab} H^{ab})$ as Ω^{-6}. We might expect this as the Bianchi identities may be written, to the linear approximation,

$$\Omega g^{ed} \frac{\partial}{\partial x^e}\left(\Omega^{-1} C_{abcd}\right) = J_{abc} .$$

Therefore if the interaction with the matter could be neglected C_{abcd} would be proportional to Ω and E_{ab}, H_{ab} to Ω^{-1}.

In the steady-state universe when μ and θ have reached their equilibrium values, $R_{ab} = \left(\tfrac{1}{2} + p\right) g_{ab}$

$$\therefore J_{abc} = R_{c[a;b]} - \tfrac{1}{6} g_{c[a} R_{;b]}$$
$$= 0$$

Thus the interaction of the "C" field with gravitational radiation is equal and opposite to that of the matter. There is then no net interaction, and E_{ab} and H_{ab} decrease as Ω^{-1}.

The "energy" $\frac{1}{2}(E_{ab}E^{ab} + H_{ab}H^{ab})$ depends on second derivatives of the metric. It is therefore proportional to the frequency squared times the energy as measured by the energy momentum pseudo-tensor, in a local co-moving Cartesian coordinate system which depends only on first derivatives. Since the frequency will be inversely proportional to Ω, the energy measured by the pseudo-tensor will be proportional to Ω^{-4} as for other rest mass zero fields.

9. Absorption of Gravitational Waves

As we have seen, gravitational waves are not absorbed by a perfect fluid. Suppose however there is a small amount of viscosity. We may represent this by the addition of a term $\lambda \sigma_{ab}$ to the energy-momentum tensor, where λ is the coefficient of viscosity (Ehlers,[17]).

Since $T_{ab}{}^{;b} = 0$

we have

$$\dot{\mu} + (\mu + p)\theta - 2\lambda\sigma^2 = 0 \tag{25}$$

$$(\mu + p)\dot{u}_a + p_b h^b{}_a + \lambda \sigma_{cb}{}^{;b} h^c{}_a = 0 \tag{26}$$

Equations (15) (16) become

$$\dot{E}_{ab} + E_{ab}\theta + h^{f}{}_{(a}\eta_{b)cde}\, u^{c} H_{f}{}^{d;e} = -\tfrac{1}{2}(\mu + p)\sigma_{ab} - \tfrac{1}{2}\lambda\left(E_{ab} - \tfrac{1}{3}\sigma_{ab}\theta\right), \qquad (27)$$

$$\dot{H}_{ab} + H_{ab}\theta - h^{f}{}_{(a}\eta_{b)cde}\, u^{c} E_{f}{}^{d;e} = -\tfrac{1}{2}\lambda H_{ab}. \qquad (28)$$

The extra terms on the right of equations (27), (28) are similar to conduction terms in Maxwell's equations and will cause the wave to decrease by a factor $e^{-\frac{\lambda}{2}t}$. Neglecting expansion for the moment, suppose we have a wave of the form,

$$E_{ab} = {}_{0}E_{ab}\, e^{i\nu\tau}.$$

This will be absorbed in a characteristic time $2/\lambda$ independent of frequency. By (25) the rate of gain of rest mass energy of the matter will be $2\lambda\sigma^{2}$ which by (19) will be $2\lambda\ {}_{0}E^{2}\nu^{-2}$. Thus the available energy in the wave is $4\ {}_{0}E^{2}\nu^{-2}$. This confirms that the density of available energy of gravitational radiation will decrease as Ω^{-4} in an expanding universe. From this we see that gravitational radiation behaves in much the same way as other radiation fields. In the early stages of an evolutionary universe when the temperature was very high we might expect an equilibrium to be set up between black-body electromagnetic radiation and black-body gravitational radition. Since they both have two polarisations their energy densities should be equal. As the universe expanded they would both cool adiabatically at the same rate. As we know the temperature of black-body extragalactic electromagnetic radiation is less than $5^{\circ}K$, the temperature of the black-body gravitational radiation must be also less than this which would be absolutely undetectable. Now the energy of gravitational radiation does not contribute to the ordinary energy momentum tensor T_{ab}. Nevertheless it will have an active gravitational effect. By the expansion equation,

$$\dot{\theta} = -\tfrac{1}{3}\theta^2 - 2\sigma^2 - \tfrac{1}{2}(\mu + 3p) .$$

For incoherent gravitational radiation at frequency ν,

$$\sigma^2 = {}_{0}E^2 \nu^{-2}$$

But the energy density of the radiation is $4\ {}_{0}E^2 \nu^{-2}$

$$\therefore \quad \dot{\theta} = -\tfrac{1}{3}\theta^2 - \tfrac{1}{2}\mu_G - \tfrac{1}{2}(\mu + 3p)$$

where μ_G is the gravitational "energy" density. Thus gravitational radiation has an active attractive gravitational effect. It is interesting that this seems to be just half that of electromagnetic radiation.

It has been suggested by Hogarth(18) and Hoyle and Narlikar(10), that there may be a connection between the absorption of radiation and the Arrow of Time. Thus in universes like the steady-state, in which all electromagnetic radiation emitted is eventually absorbed by other matter, the Absorber theory would predic retarded solutions of the Maxwell equations while in evolutionary universes in which electromagnetic radiation is not completely absorbed it would predict advanced solutions. Similarly, if one accepted this theory, one would expect retarded solutions of the Einstein equations if and only if all gravitational radiation emitted is eventually absorbed by other matter. Clearly this is so for the steady-state universe since λ will be constant. In evolutionary universes λ will be a function of time. We will obtain complete absorption if $\int \lambda \, d\tau$ diverges. Now for a gas, $\lambda \propto T^{\frac{1}{2}}$ where T is the temperature. For a monatomic gas, $T \propto \Omega^{-2}$, therefore the integral will diverge (just). However the expression used for viscosity assumed that the mean free path of the atoms was small compared to the scale of the disturbance. Since the mean free path $\propto \mu^{-1} \propto \Omega^{-3}$ and the wavelength $\propto \Omega^{-1}$, the mean free path will eventually be greater than the wavelength and so the effective viscosity will decrease more rapidly than Ω^{-1}. Thus there will not be complete absorption and the theory would not predict retarded solutions. However this is slightly academic since gravitational radiation has not yet been detected, let alone investigated to see whether it corresponds to a retarded or advanced solution.

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CHAPTER 3

Gravitational Radiation In An Expanding Universe

Gravitational radiation in empty asymptotically flat space has been examined by means of asymptotic expansions by a number of authors.(1-4) They find that the different components of the outgoing radiation field "peel off", that is, they go as different powers of the affine radial distance. If one wishes to investigate how this behaviour is modified by the presence of matter, one is faced with a difficulty that does not arise in the case of, say, electromagnetic radiation in matter. For this one can consider the radiation travelling through an infinite uniform medium that is static apart from the disturbance created by the radiation. In the case of gravitational radiation this is not possible. For, if the medium were initially static, its own self gravitation would cause it to contract in on itself and it would cease to be static. Hence one is forced to investigate gravitational radiation in matter that is either contracting or expanding.

As in Chapter 2, we identify the Weyl or conformal tensor C_{abcd} with the free gravitational field and the Ricci-tensor R_{ab} with the contribution of the matter to the curvature. Instead of considering gravitational radiation in asymptotically flat space, that is, space that approaches flat space at large radial distances, we consider it in asymptotically conformally flat space. As it is only conformally flat, the Ricci-tensor and the density of matter need not be zero.

To avoid essentially non-gravitational phenomena such as sound waves, we will consider gravitationalradiation travelling through dust. It was shown in Chapter 2 that a conformally flat universe filled with dust must have one of the metrics:

(a) $$ds^2 = \Omega^2\left(dt^2 - d\rho^2 - \sin^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$
$$\Omega = A(1 - \cos t) \qquad (1.1)$$

(b) $$ds^2 = \Omega^2\left(dt^2 - d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$
$$\Omega = \tfrac{1}{2}At^2 \qquad (1.2)$$

(c) $$ds^2 = \Omega^2\left(dt^2 - d\rho^2 - \sinh^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$
$$\Omega = A(\cosh t - 1) \qquad (1.3)$$

Type(a) represents a universe in which the matter expands from the initial singularity with insufficient energy to reach infinity and so falls back again to another singularity. It is therefore unsuitable for a discussion of gravitational radiation by a method of asymptotic expansions since one cannot get an infinite distance from this source.

Type (b) is the Einstein-De Sitter universe in which the matter has just sufficient energy to reach infinity. It is thus a special case. D. Norman [5] has investigated the "peeling off" behaviour in this case using Penrose's conformal technique [6]. He was however forced to make certain assumptions about the movement of the matter which will be shown to be false. Moreover, he was misled by the special nature of the Einstein-De Sitter universe in which affine and luminosity distances differ. Another reason for not considering radiation in the Einstein-De Sitter universe is that it is unstable. The passage of a gravitational wave will cause it to contract again eventually and develop a singularity.

We will therefore consider radiation in a universe of type (c) which corresponds to the general case where the matter is expanding with more than enough energy to avoid contracting again.

2. The Newman-Penrose Formalism

We employ the notation of Newman and Penrose. [3] A tetrad of null vectors, $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$ is introduced gravitational radiation by a method of asymptotic expansions since one cannot get an infinite distance from this source.

Type (b) is the Einstein-De Sitter universe in which the matter has just sufficient energy to reach infinity. It is thus a special case. D. Norman [5] has investigated the "peeling off" behaviour in this case using Penrose's conformal technique [6]. He was however forced to make certain assumptions about the movement of the matter which will be shown to be false. Moreover, he was misled by the special nature of the Einstein-De Sitter universe in which affine and luminosity distances differ. Another reason for not considering radiation in the Einstein-De Sitter universe is that it is unstable. The passage of a gravitational wave will cause it to contract again eventually and develop a singularity.

We will therefore consider radiation in a universe of type (c) which corresponds to the general case where the matter is expanding with more than enough energy to avoid contracting again.

2. The Newman-Penrose Formalism

We employ the notation of Newman and Penrose.[3] A tetrad of null vectors, $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$ is introduced where: $l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = l_\mu m^\mu = n_\mu m^\mu = 0$

$$l_\mu n^\mu = 1, \qquad m_\mu \bar{m}^\mu = -1$$

we label these vectors with a tetrad index

$$\underset{a}{Z}^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu) \qquad a = 1, 2, 3, 4.$$

tetrad indices are raised and lowered with the metric

$$\eta_{ab} = \eta^{ab} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad (2.1)$$

we have,

$$g^{\mu\nu} = \eta^{ab} \underset{a}{Z}^\mu \underset{b}{Z}^\nu$$
$$= l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu \qquad (2.2)$$

Ricci rotation coefficients are defined by:

$$\underset{a}{\gamma}^{bc} = \underset{a}{Z}_{\mu;\nu} \overset{b}{Z}^\mu \overset{c}{Z}^\nu \qquad (2.3)$$

$$\gamma^{abc} = -\gamma^{bac}$$

In fact it is more convenient to work in terms of twelve complex combinations of rotation coefficients defined as follows:

$$\kappa = \gamma_{131} = L_{\mu;\nu} M^{\mu} L^{\nu}$$

$$\pi = -\gamma_{241} = -N_{\mu;\nu} \bar{M}^{\mu} L^{\nu}$$

$$\varepsilon = \tfrac{1}{2}\left(\gamma_{121} - \gamma_{341}\right) = \tfrac{1}{2}\left(L_{\mu;\nu} N^{\mu} L^{\nu} - M_{\mu;\nu} \bar{M}^{\mu} L^{\nu}\right)$$

$$\rho = \gamma_{134} = L_{\mu;\nu} M^{\mu} \bar{M}^{\nu}$$

$$\lambda = -\gamma_{244} = N_{\mu;\nu} \bar{M}^{\mu} \bar{M}^{\nu}$$

$$\alpha = \tfrac{1}{2}\left(\gamma_{124} - \gamma_{344}\right) = \tfrac{1}{2}\left(L_{\mu;\nu} N^{\mu} \bar{M}^{\nu} - M_{\mu;\nu} \bar{M}^{\mu} \bar{M}^{\nu}\right)$$

$$\beta = \tfrac{1}{2}\left(\gamma_{123} - \gamma_{343}\right) = \tfrac{1}{2}\left(L_{\mu;\nu} N^{\mu} M^{\nu} - M_{\mu;\nu} \bar{M}^{\mu} M^{\nu}\right)$$

$$\sigma = \gamma_{133} = L_{\mu;\nu} M^{\mu} M^{\nu}$$

$$\mu = -\gamma_{243} = -N_{\mu;\nu} \bar{M}^{\mu} M^{\nu}$$

$$\nu = -\gamma_{242} = -N_{\mu;\nu} \bar{M}^{\mu} N^{\nu}$$

$$\gamma = \tfrac{1}{2}\left(\gamma_{122} - \gamma_{342}\right) = \tfrac{1}{2}\left(L_{\mu;\nu} N^{\mu} N^{\nu} - M_{\mu;\nu} \bar{M}^{\mu} N^{\nu}\right)$$

$$\tau = \gamma_{132} = L_{\mu;\nu} M^{\mu} N^{\nu} \qquad (2.4)$$

3. Coordinates

Like Newman and Penrose, we introduce a null coordinate $u(=x^1)$

$$g^{\mu\nu}u_{;\mu}u_{;\nu}=0 \qquad (3.1)$$

we take $L_\mu = u_{;\mu}$. Thus L_μ will be geodesic and irrotational. This implies

$$\begin{aligned} \kappa &= 0 \\ \rho &= \bar{\rho} \\ \epsilon &= -\bar{\epsilon} \\ \tau &= \bar{\alpha}+\beta \end{aligned} \qquad (3.2)$$

we take $n^\mu, m^\mu, \bar{m}^\mu$ to be parallelly transported along L^μ . This gives

$$\pi = \epsilon = 0 \qquad (3.3)$$

As a second coordinate we take an affine parameter $r(=x^2)$ long the geodesics L_μ

$$r_{;\mu}L^\mu = 1 \qquad (3.4)$$

x^3 and x^4 are two coordinates that label the geodesic in the surface u = const.

$$x^3_{;\mu}L^\mu = x^4_{;\mu}L^\mu = 0 \qquad (3.5)$$

Thus $g^{\mu\nu}$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & g^{22} & g^{23} & g^{24} \\ 0 & g^{23} & g^{33} & g^{34} \\ 0 & g^{24} & g^{34} & g^{44} \end{bmatrix} \qquad (3.6)$$

In these coordinates $L_\mu = \delta^1_\mu,\ L^\mu = \delta^\mu_2$

since $L_\mu n^\mu = 1,\ L_\mu m^\mu = 0$

$$\begin{aligned} m^\mu &= \omega\delta^\mu_2 + \xi^i\delta^\mu_i \qquad\qquad i = 3,4 \qquad (3.7) \\ n^\mu &= \delta^\mu_1 + U\delta^\mu_2 + X^i\delta^\mu_i \end{aligned}$$

The Field Equations

We may calculate the Ricci and Weyl tensor components from the relations

$$R^{abcd} = \gamma^{abc;d} - \gamma^{abd;c} + \gamma^{ad}{}_{e}\gamma^{ebc} - \gamma^{ac}{}_{e}\gamma^{ebd} + \gamma^{abe}\left(\gamma_{e}{}^{cd} - \gamma_{e}{}^{dc}\right) \qquad (3.8)$$

$$R_{abcd} = C_{abcd} + \eta_{a[d}R_{c]b} + \eta_{b[c}R_{d]a} + \frac{R}{3}\eta_{a[c}\eta_{d]b} \qquad (3.9)$$

Using the combinations of rotation coefficients already defined and with $\kappa = \pi = \varepsilon = 0$ we have

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \phi_{00} \qquad (3.10)$$

$$D\sigma = 2\rho\sigma + \Psi_0 \qquad (3.11)$$

$$D\tau = \tau\rho + \bar{\tau}\sigma + \Psi_1 + \phi_{01} \qquad (3.12)$$

$$D\alpha = \alpha\rho + \beta\bar{\sigma} + \phi_{10} \qquad (3.13)$$

$$D\beta = \beta\rho + \alpha\sigma + \Psi_1 \qquad (3.14)$$

$$D\gamma = \tau\alpha + \bar{\tau}\beta + \Psi_2 - \Lambda + \phi_{11} \qquad (3.15)$$

$$D\lambda = \lambda\rho + \mu\bar{\sigma} + \phi_{20} \qquad (3.16)$$

$$D\mu = \mu\rho + \lambda\sigma + \Psi_2 + 2\Lambda \qquad (3.17)$$

$$D\nu = \tau\lambda + \bar{\tau}\mu + \Psi_3 + \phi_{21} \qquad (3.18)$$

$$\Delta\lambda - \bar{\delta}\nu = 2\alpha\nu + (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda - \psi_4 \qquad (3.19)$$

$$\delta\rho - \bar{\delta}\sigma = (\beta + \bar{\alpha})\rho + (\bar{\beta} - 3\alpha)\sigma - \psi_1 + \phi_{01} \qquad (3.20)$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma - 2\alpha\beta + \alpha\bar{\alpha} + \beta\bar{\beta} - \psi_2 + \Lambda + \phi_{11} \qquad (3.21)$$

$$\delta\lambda - \bar{\delta}\mu = (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \psi_3 + \phi_{21} \qquad (3.22)$$

$$\delta\nu - \Delta\mu = \gamma\mu - 2\nu\beta + \bar{\gamma}\mu + \mu^2 + \lambda\bar{\lambda} + \phi_{22} \qquad (3.23)$$

$$\delta\gamma - \Delta\beta = \tau\mu - \sigma\nu + (\mu - \gamma + \bar{\gamma})\beta + \bar{\lambda}\alpha + \phi_{12} \qquad (3.24)$$

$$\delta\tau - \Delta\sigma = 2\tau\beta + (\bar{\gamma} + \mu - 3\gamma)\sigma + \bar{\lambda}\rho + \phi_{02} \qquad (3.25)$$

$$\Delta\rho - \bar{\delta}\tau = (\gamma + \bar{\gamma} - \bar{\mu})\rho - 2\alpha\tau - \lambda\sigma - \psi_2 - 2\Lambda \qquad (3.26)$$

$$\Delta\alpha - \delta\gamma = \rho\nu - \tau\lambda - \lambda\beta + (\bar{\gamma} - \gamma - \bar{\mu})\alpha - \psi_3 \qquad (3.27)$$

where $$D = L^{\mu}\nabla_{\mu} = \frac{\partial}{\partial r} \qquad (3.28)$$

$$\Delta = n^{\mu}\nabla_{\mu} = V\frac{\partial}{\partial r} + \frac{\partial}{\partial u} + X^{i}\frac{\partial}{\partial x^{i}} \qquad (3.29)$$

$$\delta = M^{\mu}\nabla_{\mu} = \omega\frac{\partial}{\partial r} + \xi^{i}\frac{\partial}{\partial x^{i}} \qquad (3.30)$$

$$\phi_{00} = -\tfrac{1}{2}\underset{11}{R} = \bar{\phi}_{00} \qquad (3.31)$$

$$\phi_{11} = -\tfrac{1}{4}\left(\underset{12}{R} + \underset{34}{R}\right) = \bar{\phi}_{11} \qquad (3.32)$$

$$\phi_{01} = -\tfrac{1}{2}\underset{13}{R} = \bar{\phi}_{10} \qquad (3.33)$$

$$\phi_{12} = -\tfrac{1}{2}\underset{23}{R} = \bar{\phi}_{21} \qquad (3.34)$$

$$\phi_{02} = -\tfrac{1}{2}\underset{33}{R} = \bar{\phi}_{20} \qquad (3.35)$$

$$\phi_{22} = -\tfrac{1}{2}\underset{22}{R} = \bar{\phi}_{22} \qquad (3.36)$$

$$\Lambda = \frac{R}{24} \qquad (3.37)$$

$$\Psi_0 = -\underset{1313}{C} = -C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma m^\delta \qquad \underline{(3.38)}$$

$$\Psi_1 = -\underset{1213}{C} = -C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma m^\delta \qquad \underline{(3.39)}$$

$$\Psi_2 = -\frac{1}{2}\left(\underset{1212}{C} + \underset{1234}{C}\right) = -\frac{1}{2} C_{\alpha\beta\gamma\delta}$$
$$\times\left(l^\alpha n^\beta l^\gamma n^\delta + l^\alpha n^\beta m^\gamma \bar{m}^\delta\right) \qquad \underline{(3.40)}$$

$$\Psi_3 = \underset{1224}{C} = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta n^\gamma \bar{m}^\delta \qquad \underline{(3.41)}$$

$$\Psi_4 = -\underset{2424}{C} = -C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta \qquad \underline{(3.42)}$$

Expressing the rotation coefficients in terms of the metric, we have:

$$D\xi^i = \rho\xi^i + \sigma\bar{\xi}^i \qquad (3.43)$$

$$D\omega = \rho\omega + \sigma\bar{\omega} - (\bar{\alpha} + \beta) \qquad (3.44)$$

$$DX^i = \tau\bar{\xi}^i + \bar{\tau}\xi^i \qquad (3.45)$$

$$DU = \tau\bar{\omega} + \bar{\tau}\omega - (\gamma + \bar{\gamma}) \qquad (3.46)$$

$$\delta X^i - \Delta\xi^i = (\mu + \bar{\gamma} - \gamma)\xi^i + \bar{\lambda}\bar{\xi}^i \qquad (3.47)$$

$$\bar{\delta}\xi^i - \delta\bar{\xi}^i = (\bar{\beta} - \alpha)\xi^i + (\bar{\alpha} - \beta)\bar{\xi}^i \qquad (3.48)$$

$$\delta\bar{\omega} - \bar{\delta}\omega = (\bar{\beta} - \alpha)\omega + (\bar{\alpha} - \beta)\bar{\omega} + (\mu - \bar{\mu}) \qquad (3.49)$$

$$\delta U - \Delta\omega = (\mu + \bar{\gamma} - \gamma)\omega + \bar{\lambda}\bar{\omega} - \bar{\nu} \qquad (3.50)$$

As in Chapter 2 we use the Bianchi identities as field equations for the Weyl tensor. In the Newman-Penrose formalism they may be written:

(I am indebted to R. G. McLenaghan for these)

$$\bar{\delta}\Psi_0 - D\Psi_1 + D\phi_{01} - \delta\phi_{00} = 4\alpha\Psi_0 - 4\rho\Psi_1 - (2\bar{\alpha} + 2\beta)\phi_{00} + 2\rho\phi_{01} + 2\sigma\phi_{10} \qquad (3.51)$$

$$\Delta\Psi_0 - \delta\Psi_1 + D\phi_{02} - \delta\phi_{01} = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 - \bar{\lambda}\phi_{00} - 2\beta\phi_{01} + 2\sigma\phi_{11} + \rho\phi_{02} \qquad (3.52)$$

$$3(\bar{\delta}\Psi_1 - D\Psi_2) + 2(D\phi_{11} - \delta\phi_{10}) + \bar{\delta}\phi_{01} - \Delta\phi_{00} = 3\lambda\Psi_0 - 9\rho\Psi_2 + 6\alpha\Psi_1 + (\bar{\mu} - 2\mu - 2\gamma - 2\bar{\gamma})\phi_{00} + (2\alpha + 2\bar{\tau})\phi_{01} + 2(\tau - 2\bar{\alpha})\phi_{10} + 2\rho\phi_{11} + 2\sigma\phi_{20} - \bar{\sigma}\phi_{02} \qquad (3.53)$$

$$3(\Delta\Psi_1 - \delta\Psi_2) + 2(D\phi_{12} - \delta\phi_{11}) + (\bar{\delta}\phi_{02} - \Delta\phi_{01}) = 3\nu\Psi_0 + 6(\gamma - \mu)\Psi_1 - 9\tau\Psi_2 + 6\sigma\Psi_3 - \bar{\nu}\phi_{00} + 2(\bar{\mu} - \mu - \gamma)\phi_{01} - 2\lambda\phi_{10} + 2\tau\phi_{11} + (2\alpha + \bar{\tau} - 2\bar{\beta})\phi_{02} + 2\sigma\phi_{21} \qquad (3.54)$$

$$3(\bar{\delta}\Psi_2 - D\Psi_3) + (D\phi_{21} - \delta\phi_{20}) + 2(\bar{\delta}\phi_{11} - \Delta\phi_{10}) = 6\lambda\Psi_1 - 6\rho\Psi_3 - 2\nu\phi_{00} + 2\lambda\phi_{01} + 2(\bar{\mu} - \mu - 2\bar{\gamma})\phi_{10} + 4\bar{\tau}\phi_{11} + (2\beta + 2\tau - 2\bar{\alpha})\phi_{20} - 2\bar{\sigma}\phi_{12} \qquad (3.55)$$

$$3(\Delta\psi_2 - \delta\psi_3) + (D\phi_{22} - \delta\phi_{21}) + 2(\bar{\delta}\phi_{12} - \Delta\phi_{11})$$
$$= 6\nu\psi_1 - 9\mu\psi_2 + 6(\beta - \tau)\psi_3 + 3\sigma\psi_4$$
$$- 2\nu\phi_{01} - 2\bar{\nu}\phi_{10} + 2(2\bar{\mu} - \mu)\phi_{11} + 2\lambda\phi_{02}$$
$$- \bar{\lambda}\phi_{20} + 2(\bar{\tau} - 2\bar{\beta})\phi_{12} + 2(\beta + \tau)\phi_{21} - \rho\phi_{22} \qquad (3.56)$$

$$\bar{\delta}\psi_3 - D\psi_4 + \bar{\delta}\phi_{21} - \Delta\phi_{20} = 3\lambda\psi_2 - 2\alpha\psi_3 - \rho\psi_4$$
$$- 2\nu\phi_{01} + 2\lambda\phi_{11} + (2\gamma - 2\bar{\gamma} + \bar{\mu})\phi_{20}$$
$$+ 2(\bar{\tau} - \alpha)\phi_{21} - \bar{\sigma}\phi_{22} \qquad (3.57)$$

$$\Delta\psi_3 - \delta\psi_4 + \bar{\delta}\phi_{22} - \Delta\phi_{21} = 3\nu\psi_2 - 2(\gamma + 2\mu)\psi_3$$
$$+ (4\beta - \tau)\psi_4 - 2\nu\phi_{11} - \bar{\nu}\phi_{20} + 2\lambda\phi_{12}$$
$$+ 2(\gamma + \bar{\mu})\phi_{21} + (\bar{\tau} - 2\bar{\beta} - 2\alpha)\phi_{22} \qquad (3.58)$$

$$D\phi_{12} - \delta\phi_{11} - \bar{\delta}\phi_{02} + \Delta\phi_{01} + 3\delta\Lambda = (2\gamma - \mu - 2\bar{\mu})\phi_{01} + \bar{\nu}\phi_{00}$$
$$- \bar{\lambda}\phi_{10} - 2\tau\phi_{11} + (2\bar{\beta} - 2\alpha - \bar{\tau})\phi_{02} + 3\rho\phi_{12} + \sigma\phi_{21} \qquad (3.59)$$

$$D\phi_{11} - \delta\phi_{10} - \bar{\delta}\phi_{01} + \Delta\phi_{00} + 3D\Lambda = (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})\phi_{00} - 2(\alpha + \bar{\tau})\phi_{01}$$
$$- 2(\bar{\alpha} + \tau)\phi_{10} + 4\rho\phi_{11} + \bar{\sigma}\phi_{02} + \sigma\phi_{20} \qquad (3.60)$$

$$D\phi_{22} - \delta\phi_{21} - \bar{\delta}\phi_{12} + \Delta\phi_{11} + 3\Delta\Lambda = \nu\phi_{01} + \bar{\nu}\phi_{10} - 2(\mu + \bar{\mu})\phi_{11}$$
$$- \lambda\phi_{02} - \bar{\lambda}\phi_{20} + (2\bar{\beta} - \bar{\tau})\phi_{12} + (2\beta - \tau)\phi_{21}$$
$$+ 2\rho\phi_{22} \qquad (3.61)$$

4. The Undisturbed Metric

The undisturbed metric may be written

$$ds^2 = \Omega^2(dt^2 - d\rho^2 - \sinh^2\rho(d\theta^2 + \sin^2 d\phi^2))$$

$$\Omega = A(\cosh t - 1)$$

put $u = t - \rho$

then $$ds^2 = \Omega^2\left[-du^2 + 2du\,dt - \sinh^2(t-u)(d\theta^2 + \sin^2\theta d\phi^2)\right] \qquad (4.1)$$

u is a null coordinate

To calculate r, the affine parameter, we note that t is an affine parameter for the metric within the square brackets. Therefore $r = \int \Omega^2 dt + B(u, \theta, \phi)$ (4.2) will be an affine parameter for (4.1)

B is constant along the null geodesic. Normally it would be taken so that $r = 0$ when $t = u$. However, in our case it will be more convenient to make it zero and define r as

$$r = \int_0^t \Omega^2 dt' \qquad (4.3)$$

This means that surfaces of constant r are surfaces of constant t. This may seem rather odd, but it should be pointed out that the choice of B will not affect the asymptotic dependence of quantities. That is, if

$$f = O(r^{-n})$$

Then

$$f = O(r'^{-n}), \quad r' = r + B$$

It proves easier to perform the calculations with this choice of r but all results could be transformed back to a more normal coordinate system.

From (4.3) $\quad r = A^2\left[\frac{1}{4}\sinh 2t - 2\sinh t + \frac{3}{2}t\right] \qquad (4.4)$

The matter in the universe is assumed to be dust so its energy tensor may be written

$$T_{ab} = \mu V_a V_b \qquad (4.5)$$

For the undisturbed case, from Chapter 2

$$\mu = \frac{6A}{\Omega^3}$$

$$V_a = \Omega t_{;a} \qquad V_a V^a = 1 \qquad (4.6)$$

Now

$$\Omega = \sqrt{2}\,s + A - \frac{3A^2}{2\sqrt{2}}\frac{\log s}{s} + O(s^{-1}) \qquad (4.7)$$

where $s^2 = r$

Therefore if we try to expand μ as a series in power of s the result will be very messy and will involve terms of the form $\frac{\log^m s}{s^n}$ *

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*It should be pointed out that the expansions used will only be assumed to be valid asymptotically. They will not be assumed to converge at finite distances nor will the quantities concerned be assumed analytic. (see A. Erdelyi: Asymptotic Expansions - Dover

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This does not invalidate it as an asymptotic expansion but it makes it tedious to handle. For convenience therefore, we will perform the expansions in terms of $\Omega(r)$ which will be defined in general as the same function of r as it is in the undisturbed case. That is

$$\Omega = A(\cosh t - 1)$$

where
$$r = A^2\left[\tfrac{1}{4}\sinh 2t - 2\sinh t + \tfrac{3}{2}t\right] \qquad (4.8)$$

then

$$\frac{d\Omega}{dr} = \frac{\sqrt{1+\frac{2A}{\Omega}}}{\Omega}$$

$$= \frac{1}{\Omega}\left[1 + \frac{A}{\Omega} - \frac{A^2}{2\Omega^2} + \frac{A^3}{2\Omega^3} - \frac{5A^4}{8\Omega^4} + \frac{7A^5}{8\Omega^5} \cdots\right] (4.9)$$

For the third and fourth coordinates it is more convenient to use stereographic coordinates than spherical polars.

Since the matter is dust its energy-momentum tensor and hence the Ricci-tensor have only four independent components. We will take these as $\Lambda, \phi_{00}, \phi_{01}$. (Since ϕ_{01} is complex it represents two components) In terms of these the other components of the Ricci-tensor may be expressed as:

$$\phi_{11} = 3\Lambda + \frac{\phi_{01}\bar{\phi}_{01}}{\phi_{00}}$$

$$\phi_{22} = \frac{36\Lambda^2}{\phi_{00}}\left(1 + \frac{\phi_{01}\bar{\phi}_{01}}{6\Lambda\phi_{00}}\right)^2$$

$$\phi_{12} = \bar{\phi}_{21} = \frac{6\Lambda\phi_{01}}{\phi_{00}}\left(1 + \frac{\phi_{01}\bar{\phi}_{01}}{6\Lambda\phi_{00}}\right)^2$$

$$\phi_{02} = \bar{\phi}_{20} = \frac{\phi_{01}{}^2}{\phi_{00}} \qquad (4.10)$$

For the undisturbed universe with the coordinate system given:

$$\Lambda = \frac{\mu}{24} = \frac{A}{4\Omega^3}$$

$$\phi_{00} = \frac{3A}{\Omega^5}$$

$$\phi_{11} = \frac{3A}{4\Omega^3}$$

$$\phi_{22} = \frac{3A}{4\Omega}$$

$$\phi_{01} = \phi_{02} = 0 \qquad (4.11)$$

Using these values and the fact that in the undisturbed universe all the ψ's are zero, we may integrate equations (3. 10-50) to find the values of the spin coefficients for the unperturbed universe:

$$\rho = -\frac{2}{\Omega^2} - \frac{A}{\Omega^3} + \left(\frac{A}{2}^2 - \frac{A^2}{2}e^{2u}\right)\Omega^{-4} - A^3\left(\frac{1}{2} - e^{2u}\right)\Omega^{-5} + A^4\left(\frac{5}{8} - \frac{7}{4}e^{2u} - \frac{1}{8}e^{4u}\right)\Omega^{-6} \ldots$$

$$\sigma = \tau = \omega = \nu = \lambda = X^i = 0$$

$$\xi^3 = \frac{Se^u}{\Omega^2} - \frac{ASe^u}{\Omega^3} + A^4\left(5SA^u + Se^{3u}\right)\Omega^{-4} + \ldots$$

$$\alpha = -\bar{\beta} = \frac{\frac{1}{2}\bar{\nabla} S e^{u}}{\Omega^2} - \frac{\frac{1}{2} A \bar{\nabla} S e^{u}}{\Omega^3} + \ldots$$

$$\mu = \frac{A}{2\Omega} - \frac{A^2}{4}(1 + e^{2u})\Omega^{-2} + \frac{A^3}{4}(1 + 2e^{2u})\Omega^{-3} + \ldots$$

$$\gamma = -\frac{1}{2} - \frac{A}{2\Omega} + \frac{A^2}{4\Omega^2} - \frac{A^3}{4\Omega^3} + \ldots$$

$$U = \frac{1}{2}\Omega^2 \qquad (4.12)$$

5. Boundary Conditions

We wish to consider radiation in a universe that asymptotically approaches the undisturbed universe given above. Φ_{00} and Λ will then have the values given above plus terms of smaller order. To determine this order and the order of Φ_{01} and Ψ_0, there are two ways in which we may proceed. We may take the smallest orders that will permit radiation, that is $\Psi_4 = O(r^{-1})$. Larger order terms than these in Φ_{00}, Λ and Φ_{01} turn out to have their u derivatives dependent only on themselves and not on the r^{-1} coefficient of Ψ_4, the radiation field. They are thus disturbances not produced by the radiation field and will not be considered.

Alternatively we may proceed by a method of successive approximations. We take the undisturbed values of the spin coefficients and use them to solve the Bianchi Identities as field equations for the conformal tensor using the flat space boundary condition that $\psi_0 = O(r^{-5})$. Then substituting these ψ's in equations (3.10 - 27) calculate the disturbances induced in the spin coefficients and substituting these back in the Bianchi Identities, calculate the disturbances in the ψ's. Further iteration does not affect the orders of the disturbances.

Both these methods indicate that the boundary conditions should be:

$$\Lambda = \frac{A}{4\Omega^3} + O(\Omega^{-7}) \qquad (5.1)$$

$$\phi_{00} = \frac{3A}{\Omega^5} + O(\Omega^{-9}) \qquad (5.2)$$

$$\phi_{01} = O(\Omega^{-7}) \text{ (see next section)} \qquad (5.3)$$

$$\psi_0 = O(\Omega^{-7}) \qquad (5.4)$$

We also assume "uniform smoothness", that is:

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\cdots\cdots\frac{\partial}{\partial x^l}\Lambda = O(\Omega^{-7})$$

$$\frac{\partial}{\partial\Omega}\Lambda = \frac{-3A}{4\Omega^4} + O(\Omega^{-8})$$

etc.../

It will be shown that if these boundary conditions hold on one hypersurface (u = const.) they will hold on succeeding hypersurfaces and that these conditions are the most severe to permit radiation.

6 Integration

As Newman and Penrose, we begin by integrating the equations (3. 10 &11)

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \phi_{00}$$
$$D\sigma = 2\rho\sigma + \psi_0$$

where

$$\phi_{00} = \frac{3A}{4\sqrt{2}\, r^{\frac{5}{2}}} + O(r^{-3})$$

$$\psi_0 = \frac{\psi_0^0}{8\sqrt{2}\, r^{\frac{7}{2}}} + O(r^{-4})$$

Let

$$P = \begin{bmatrix} \rho & \sigma \\ \bar{\sigma} & \rho \end{bmatrix} \qquad \varphi = \begin{bmatrix} \phi_{00} & \psi_0 \\ \bar{\psi}_0 & \phi_{00} \end{bmatrix}$$

then $DP = P^2 + \varphi$ (6.1)

let $P = -(DY)Y^{-1}$ (6.2)

then $D^2 Y = -\varphi Y$ (6.3)

since $\int r \varphi \, dr < \infty$

$$DY = F + O(1) \quad \text{where } F \text{ is constant} \quad (6.4)$$

$$Y = rF + O(r)$$

However $\varphi = O(r^{-\frac{5}{2}})$

therefore $D^2 Y = -r\varphi F + O(r^{-\frac{3}{2}}) = O(r^{-\frac{3}{2}})$ (6.5)

therefore $DY = F + O(r^{-\frac{1}{2}})$

$$Y = rF + O(r^{\frac{1}{2}}) + E \quad E \text{ is constant}$$

$$P = -r^{-1}I + O(r^{-\frac{3}{2}}) \quad (6.6)$$

if F is non-singular (The case F singular corresponds to asymptotically plane or cylindrical surfaces and will not be considered here).

Thus $\rho = -r^{-1} + O(r^{-\frac{3}{2}}) = -2\Omega^{-2} + O(\Omega^{-3})$

$\sigma = O(r^{-\frac{3}{2}}) = O(\Omega^{-3})$ (6.7)

Let $\rho = -2\Omega^{-2} + g\Omega^{-3}$ (6.8)

$\sigma = h\Omega^{-3}$

where $g, h, = O(1)$

Then using:

$$D = \frac{\partial}{\partial r} = \Omega^{-1}\left(1 + A\Omega^{-1} - \frac{A^2}{2}\Omega^{2} + \frac{A^3}{2}\Omega^{3} \ldots\right)\frac{\partial}{\partial \Omega}$$

$$\frac{\partial g}{\partial \Omega}\left(\Omega + O(1)\right) = -A - g + O(\Omega^{-1})$$

Integrating, $$g = \frac{1}{\Omega + O(1)}\left[a + \int \frac{(-A + O(\Omega^{-1}))\,(\Omega + O(1))}{\Omega + O(1)}\,d\Omega\right]$$

therefore $$g = -A + O\left(\frac{\log \Omega}{\Omega}\right) \qquad (6.9)$$

For h,

$$\frac{\partial h}{\partial \Omega}\left(\Omega + O(1)\right) = -h + O(\Omega^{-1})$$

therefore

$$h = O(\Omega^{-1}) \qquad (6.10)$$

Repeat the process with

$$\rho = -2\Omega^{-2} - A\Omega^{-3} + g\Omega^{-4} \qquad \underline{(6.12)}$$
$$\sigma = h\Omega^{-4}$$

where $\quad g = O(\log\Omega)$

$\quad h = O(1)$

then $\quad \frac{\partial h}{\partial\Omega}(\Omega + O(1)) = O(\Omega^{-1})$

$$h = \sigma^{0}(u, x^{i}) + O(\Omega^{-1}) \qquad \underline{(6.13)}$$

$$\frac{\partial g}{\partial\Omega}(-\Omega + O(1)) = O(\Omega^{-1}\log\Omega)$$

$$g = \rho^{0}(u, x^{i}) + O(\Omega^{-1}\log\Omega) \qquad (6.14)$$

Unlike Newman and Unti, we cannot make ρ^{0} zero by the transformation $r' = r - \rho^{0}$, since this would alter the boundary condition $\Lambda = \frac{A}{6\Omega^{3}} + \frac{\Lambda^{0}}{\Omega^{7}}$

Continuing the above process we derive:

$$\rho = -2\Omega^{-2} - A\Omega^{-3} + \rho^{0}\Omega^{-4} + (\tfrac{1}{2}A^{2} - 2A\rho^{0})\Omega^{-5}$$
$$+ \left(-\frac{5}{4}A^{4} + 4A^{2}\rho^{0} - \frac{\rho^{0\,2}}{2} - \frac{\sigma^{0}\bar{\sigma}^{0}}{2}\right)\Omega^{-6} + O(\Omega^{-7}) \qquad \underline{(6.15)}$$

$$\sigma^{0} = \sigma^{0}\Omega^{-4} - (2A\sigma^{0} + \Psi^{0}_{0})\Omega^{-5} + \ldots \qquad \underline{(6.16)}$$

To determine the asymptotic behaviour of $\psi_1, \alpha, \beta, \xi^i$ and ω we use the lemma proved by Newman and Penrose:

The $n \times n$ matrix B and the column vector b are given functions of x such that:

$$B = O(x^{-2}) \ , \quad b = O(x^{-2}) \tag{6.17}$$

The $n \times n$ matrix A is independent of x and has no eigenvalue with positive real part. Any eigenvalue with vanishing real part is regular. Then all solutions of:

$$\frac{\partial}{\partial x} y = (A x^{-1} + B) y + b \tag{6.18}$$

are bounded as $x \to \infty$. y is a column vector.

For reasons to be explained below, we will assume for the moment that

$$\begin{aligned} \phi_{01} &= O(\Omega^{-5}) \\ \frac{\partial}{\partial \Omega} \phi_{01} &= O(\Omega^{-6}) \\ \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \cdots \phi_{01} &= O(\Omega^{-5}) \end{aligned} \tag{6.19}$$

We take as y the column vector

$$[\Omega^5 \psi_1, \Omega^2 A, \Omega^2 \bar{A}, \Omega^2 \beta, \Omega^2 \xi^3, \Omega^2 \bar{\xi}^3, \Omega^2 \xi^4, \Omega \bar{\xi}^4, \omega, \bar{\omega}] \tag{6.20}$$

By equations 3.51, 3.13, 3.14, 3.43, 3.44

$$A = \begin{vmatrix} -3 & 0 & 6A & 6A & 0 & 0 & 0 & 0 & 0 & 15A & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & & & & & & 0 \\ 0 & & & & & & & & & & 0 \\ 0 & & & & & & & & & & 0 \\ 0 & & & & & & & & & & 0 \\ 0 & & & & & & & & & & 0 \\ 0 & & & & & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -2 \end{vmatrix} \qquad (6.21)$$

B and b are $O(\Omega^{-1})$ expressions involving ρ, σ, ψ_0, $\frac{\partial}{\partial\Omega}\psi_0$, $\frac{\partial}{\partial x^i}\psi_0$, ϕ_{01} and $\frac{\partial}{\partial\Omega}\phi_{01}$

Thus

$$\begin{aligned} \psi_1 &= O(\Omega^{-5}) \\ \alpha, \beta, \xi^3, \xi^4 &= O(\Omega^{-2}) \\ \omega &= O(1) \end{aligned} \qquad (6.22)$$

Since

$$\tau = \bar{\alpha} + \beta$$

$$\tau = O(\Omega^{-2})$$

Using this we integrate equation (3.12) by the same method as above. We obtain

$$\tau = \tau^{\circ}(u, x^{i})\Omega^{-2} + O(\Omega^{-3}) \qquad (6.23)$$

We may make a null rotation of the tetrad on each null geodesic

$$\begin{aligned} l'_{\mu} &= l_{\mu} \\ n'_{\mu} &= n_{\mu} + a\bar{m}_{\mu} + \bar{a}m_{\mu} + a\bar{a}\, l_{\mu} \qquad (6.24) \\ m'_{\mu} &= m_{\mu} + a l_{\mu} \end{aligned}$$

a is constant alone the geodesic since the tetrad is parallelly transported.

By taking $a = -\frac{1}{2}\tau^{\circ}$ we may make $\tau'^{\circ} = 0$

Under a null rotation

$$\phi'_{01} = \phi_{01} + a\,\phi_{00}$$

Thus until we have specified the null rotation we cannot impose a boundary condition on ϕ_{01} more severe than $\phi_{01} = O(\Omega^{-5})$. We will specify the null rotation by $\tau^{\circ} = 0$ and in that tetrad system will impose the boundary condition that $\phi_{01} = O(\Omega^{-7})$ and is uniformly smooth.

Then by using this condition on ϕ_{01} and $\tau = O(\Omega^{-3})$, by equation (3.44)

$$\omega = O(\Omega^{-1})$$

using this in equation (3.51)

$$\Psi_1 = O(\Omega^{-6})$$

then by equation (3.12) $\tau = O(\Omega^{-4})$

putting this back in equation (3.44)

$$\omega = O(\Omega^{-2} \log \Omega)$$

by equation (3.51)

$$\Psi_1 = O(\Omega^{-7} \log \Omega)$$

by equation (3.12)

$$\tau = O(\Omega^{-5} \log \Omega)$$

by equation (3.44)

$$\omega = \omega^0 \Omega^{-2} + O(\Omega^{-3} \log \Omega)$$

by equation (3.51)

$$\Psi_1 = O(\Omega^{-7}) \qquad (6.25)$$

by equation (3.12)

$$\tau = O(\Omega^{-5}) \qquad (6.26)$$

by equation (3.44)

$$\omega = \omega^0 \Omega^{-2} + O(\Omega^{-3}) \qquad (6.27)$$

By differentiating the equations used with respect to x^i, one may show that $\Psi_1, \alpha, \beta, \tau, \xi^i, \omega$ are uniformly smooth.

Adding equations 3.53 and: 3.60 :-

$$\bar{\delta}\Psi_1 - D\Psi_2 + D\Phi_{11} - \delta\Phi_{01} + D\Lambda = \lambda\Psi_0 - 3\rho\Psi_2 + 2\alpha\Psi_1 - \mu\Phi_{00} - 2\bar{\alpha}\Phi_{10} + 2\rho\Phi_{11} + \sigma\Phi_{20} \qquad (6.28)$$

We may use the lemma again with y

$$y = \begin{bmatrix} \Omega^4 \psi_2 \\ \Omega^2 \lambda \\ \Omega \mu \end{bmatrix}$$

By equations 3.16; 3.17; 6.28.

$$A = \begin{bmatrix} -2 & 0 & 3A \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

B and b are $O(\Omega^{-2})$

Therefore

$$\psi_2 = O(\Omega^{-4})$$

$$\lambda = O(\Omega^{-2}) \qquad (\underline{6.29})$$

$$\mu = O(\Omega^{-1})$$

and are uniformly smooth

From (6.29) and (3.17), we may show

$$\mu = \frac{A}{2}\Omega^{-1} + O(\Omega^{-2}\log\Omega)$$

using this in (6.28)

$$\psi_2 = O(\Omega^{-5}\log\Omega)$$

then by (3.17)

$$\mu = \frac{A}{2}\Omega^{-1} + \mu^{\circ}(u, x^i)\Omega^{-2} + O(\Omega^{-3}\log\Omega) \qquad (\underline{6.30})$$

and $\psi_2 = O(\Omega^{-5})$

Integrating the radial equations 3.13, 3.14, 3.15, 3.16, 3.43, 3.45 3.46.

$$\alpha = \alpha^{\circ}\Omega^{-2} - A\alpha^{\circ}\Omega^{-3} + \tfrac{1}{2}(3A^{2}\alpha^{\circ} - \rho^{\circ}\alpha^{\circ} + \bar{\alpha}^{\circ}\bar{\sigma}^{\circ})\Omega^{-4} + O(\Omega^{-5}) \quad (6.31)$$

$$\beta = \beta^{\circ}\Omega^{-2} - A\beta^{\circ}\Omega^{-3} + \tfrac{1}{2}(3A^{2}\beta^{\circ} - \rho^{\circ}\beta^{\circ} + \bar{\beta}^{\circ}\sigma^{\circ})\Omega^{-4} + O(\Omega^{-5}) \quad (6.32)$$

$$\gamma = \gamma^{\circ} - \frac{A}{2}\Omega^{-1} + \frac{A^{2}}{4}\Omega^{-2} + O(\Omega^{-3}) \quad (6.33)$$

$$\lambda = \lambda^{\circ}\Omega^{-2} - A\left(\lambda^{\circ} + \frac{\bar{\sigma}^{\circ}}{2}\right)\Omega^{-3} + O(\Omega^{-4}) \quad (6.34)$$

$$\xi^{i} = \xi^{i\circ}\Omega^{-2} - A\Omega^{3}\xi^{i\circ} + \tfrac{1}{2}(3A^{2}\xi^{i\circ} - \rho^{\circ}\xi^{i\circ} - \bar{\xi}^{i\circ}\sigma^{\circ})\Omega^{-4} + O(\Omega^{-5}) \quad (6.35)$$

$$X^{i} = X^{i\circ} + O(\Omega^{-5}) \quad (6.36)$$

$$U = -\tfrac{1}{2}(\gamma^{\circ} + \bar{\gamma}^{\circ})\Omega^{2} + A(1 + \gamma^{\circ} + \bar{\gamma}^{\circ})\Omega - \frac{3}{2}A^{2} \times (1 + \gamma^{\circ} + \bar{\gamma}^{\circ})\log\Omega + U^{\circ} + O(\Omega^{-1}) \quad (6.37)$$

Adding equation (3.55) + 2 x (3.59)

$$\delta\psi_{2} - D\psi_{3} + D\phi_{21} - \delta\phi_{20} + 2\bar{\delta}\Lambda = 2\lambda\psi_{1} - 2\rho\psi_{3} - \tfrac{1}{3}(\bar{\mu} + 2\mu)\phi_{10} + 2(\beta - \bar{\alpha})\phi_{20} + 2\rho\phi_{21} \quad (6.38)$$

Therefore $$\psi_3 = \psi_3^{\,0}\,\Omega^{-4} + O(\Omega^{-5}) \qquad (6.39)$$

By equation (3.18)

$$V = V^0 - \tfrac{1}{2}\psi_3^{\,0}\,\Omega^{-2} + O(\Omega^{-3}) \qquad (6.40)$$

By the orthonormality relations (2.2).

$$g^{2i} = X^i - (\xi^i\bar\omega + \bar\xi^i\omega)$$
$$= X^{i0} + O(\Omega^{-4})$$
$$g^{ij} = -(\xi^i\bar\xi^j + \bar\xi^i\xi^j) \qquad i,j = 3,4$$
$$= -(\xi^{i0}\bar\xi^{j0} + \bar\xi^{i0}\xi^{j0})(\Omega^{-4} - 2A\Omega^{-5}) + O(\Omega^{-6}) \qquad (6.41)$$

By making the coordinate transformation

$$u' = u$$
$$r' = r$$
$$x'^i = x^i + C^i(u, x^i)$$

where

$$C^3_{,1} = -X^{30}(1 + C^3_{,3}) - X^{40}C^3_{,4}$$
$$C^4_{,1} = -X^{40}(1 + C^4_{,4}) - X^{30}C^4_{,3} \qquad (6.42)$$

$X'^{i0} = 0$

We still have the coordinate freedom

$$x^i = D^i(x_j)$$

We may use this to reduce the leading term of $g^{ij}(i,j = 3,4)$ to a conformally flat metric (c.f. Newman and Unti), that is:

$$g^{ij} = -2P\bar{P}\delta^{ij}(\Omega^{-4} - 2A\Omega^{-5}) + O(\Omega^{-6}) \qquad (6.43)$$

where $$\xi^{30} = -i\xi^{40} = P(u, x^i).$$

7. Non-radial Equations

By comparing coefficients of the various powers of Ω in the non-radial equations of §3, relations between the integration constants of the radial equations may be obtained:

In equation 3.23 the term in Ω^{-1} is
$$-\frac{3}{4}A(\gamma^{\circ} + \bar{\gamma}^{\circ}) - \frac{3}{4}A$$

therefore $$\gamma^{\circ} + \bar{\gamma}^{\circ} = -1 \qquad (7.1)$$

therefore by (6.37)
$$U = \tfrac{1}{2}\Omega^{2} + U^{\circ} + O(\Omega^{-1}) \qquad (7.2)$$

In equation (3.50), the constant term is
$$V^{\circ}$$

therefore $$V^{\circ} = 0 \qquad (7.3)$$

By the Ω^{-2} term in (3.47)
$$P - P_{,i} = (\bar{\gamma}^{\circ} - \gamma^{\circ})P$$

By the Ω^{-3} term
$$P - P_{,i} = 0$$

therefore $$\gamma^{\circ} = \bar{\gamma}^{\circ} = -\frac{1}{2} \qquad (7.4).$$

$$P = S(x^i)e^u$$

if $\quad S = |S|e^{i\phi}$

By making a spatial rotation of the tetrad

$$m'^{\mu} = e^{-i\phi}m^{\mu}$$

we make S real. We take $S = \frac{1}{\sqrt{2}}\left(1+\frac{1}{4}\left(x^{3^2}+x^{4^2}\right)\right)$, the stereographic project factor for a 2-sphere.

By the Ω^{-4} term in equation (3.48)

$$\begin{aligned} \alpha^{0} &= \tfrac{1}{2}\bar{\nabla} S e^{u} \\ \beta^{0} &= -\tfrac{1}{2}\nabla S e^{u} \end{aligned} \qquad (7.5)$$

where $\quad \nabla = \dfrac{\partial}{\partial x^3} + i\dfrac{\partial}{\partial x^4}$

By the Ω^{-4} term in (3.21)

$$\tfrac{1}{2}e^{2u}(S\nabla\bar{\nabla}S + S\bar{\nabla}\nabla S) = -2\mu^{0} - \frac{A^2}{2} + (\nabla P)(\bar{\nabla}P)$$

$$\mu^{0} = -\frac{A^2}{4} - \frac{S^2}{2}\nabla\bar{\nabla}\log S e^{2u}$$

$$= -\frac{A^2}{4}\left[1 + e^{2u}\right] \qquad (7.6)$$

By $\quad (3.59)\,(3.60)\,(3.61)$

$$\frac{\partial}{\partial u}\phi_{01} = O(\Omega^{-7})$$

$$\frac{\partial}{\partial u} \phi_{00} = O(\Omega^{-7})$$

$$\frac{\partial}{\partial u} \Lambda = H\Omega^{-5} + O(\Omega^{-6} \log \Omega) \qquad (7.7)$$

where

$$H = -\frac{A^2}{8} + \frac{3U^0}{4} + \frac{\rho^0}{4} + \frac{A^2 e^{2u}}{8}$$

By making a coordinate transformation*

$$r' = r - \int_{u0}^{u} H \, du'' \qquad (7.8)$$

$$H' = 0$$

therefore

$$U^0 = \frac{A^2}{6} - \frac{A^2 e^{2u}}{6} - \frac{\rho^0}{3}$$

By the Ω^{-4} term in (3.26)

$$\rho^0_{,1} - \rho^0 + 4U^0 = -\frac{A^2}{2}(1 + e^{2u}) \qquad (7.9)$$

therefore $\rho^0_{,1} - \frac{7}{3}\rho^0 + \frac{7}{6}A^2 - \frac{1}{6}A^2 e^{2u} = 0$

$$\rho^0 = \frac{A^2}{2}\left(1 - e^{2u}\right) + C(x^i) e^{\frac{7}{3}u} \qquad (7.10)$$

---

*This transformation does not upset the boundary conditions on the hypersurface $u = u^0$

By the Ω^{-4} term in (3.25)

$$\lambda^{\circ} = \frac{1}{2}\left(\bar{\sigma}^{\circ}_{,1} - \bar{\sigma}^{\circ}\right) \qquad (7.11)$$

By the Ω^{-4} term in (3.22)

$$\Psi^{\circ}_{3} = \left(\bar{\sigma}^{\circ}_{,1} - \bar{\sigma}^{\circ}\right) e^{u} \nabla S - \frac{1}{2} e^{u} S \nabla\left(\bar{\sigma}^{\circ}_{,1} - \bar{\sigma}^{\circ}\right) \qquad (7.12)$$

By the Ω^{-2} term in (3.50)

$$2\omega^{\circ} - \omega^{\circ}_{,1} = \frac{1}{2}\bar{\Psi}^{\circ}_{3}$$

$$\therefore \omega^{\circ} = \frac{e^{u}}{4}\left(S\bar{\nabla}\sigma^{\circ} - 2\sigma^{\circ}\bar{\nabla}S\right) + K(x^{i})e^{2u} \qquad (7.13)$$

By the Ω^{-6} term in (3.20)

$$e^{u} S \nabla\rho^{\circ} + 4\omega^{\circ} - e^{u} S\bar{\nabla}\sigma^{\circ} = -2\sigma^{\circ}\bar{\nabla} S e^{u}$$

therefore $C = K = 0$ (7.14)

therefore $U^{\circ} = 0$ (7.15)

$$\rho^{\circ} = \frac{A^{2}}{2}\left(1 - e^{2u}\right) \qquad (7.16)$$

Using (7.6),(7.16), in (6.28)

$$\Psi_{2} = O(\Omega^{-6}\log\Omega)$$

Then by (3.17)

$$\mu = \frac{A}{2}\Omega^{-1} - \frac{A^{2}}{4}\left(1 + e^{2u}\right)\Omega^{-2} + \frac{A^{3}}{2}\left(\frac{1}{2} + e^{2u}\right)\Omega^{-3} + O(\Omega^{-4}\log\Omega) \qquad (7.17)$$

Using this in (6.28)

$$\Psi_{2} = O(\Omega^{-6}) \qquad (7.18)$$

By (3.51) (3.59) (3.60) (3.61)

$$\frac{\partial}{\partial u}\phi_{01} = O(\Omega^{-7})$$

$$\frac{\partial}{\partial u}\phi_{00} = O(\Omega^{-9})$$

$$\frac{\partial}{\partial u}\Lambda = O(\Omega^{-7})$$

$$\frac{\partial}{\partial u}\psi_0 = O(\Omega^{-7}) \qquad (7.19)$$

Therefore if the boundary conditions (5. 1-4) hold on one null hypersurface, they will hold on succeeding hypersurfaces.

By (3.57)

$$\psi_4 = O(\Omega^{-2}) \qquad (7.20)$$

The "peeling off" behaviour is therefore:

$$\begin{aligned} \psi_4 &= O(r^{-1}) \\ \psi_3 &= O(r^{-2}) \\ \psi_2 &= O(r^{-3}) \\ \psi_1 &= O(r^{-\frac{7}{2}}) \\ \psi_0 &= O(r^{-\frac{7}{2}}) \end{aligned} \qquad (7.21)$$

As mentioned before, this asymptotic behaviour is independent of the zero of r and will hold for any affine parameter r.

To perform the remaining integrations we will assume for definiteness:

$$
\begin{aligned}
\phi_{01} &= \phi_{01}^{0}\Omega^{-7} + \phi_{01}^{1}\Omega^{-8} + O(\Omega^{-9}) \qquad (7.22)\\
\psi_{0} &= \psi_{0}^{0}\Omega^{-7} + \psi_{0}^{1}\Omega^{-8} + O(\Omega^{-9})\\
\Lambda &= \frac{A}{4}\Omega^{-3} + \Lambda^{0}\Omega^{-7} + O(\Omega^{-8})\\
\phi_{00} &= 3A\Omega^{-5} + \phi_{00}^{0}\Omega^{-9} + O(\Omega^{-10})
\end{aligned}
$$

Then:

$$
\begin{aligned}
\rho &= -2\Omega^{-2} - A\Omega^{-3} + \frac{A^{2}}{2}(1-e^{2u})\Omega^{-4} - \frac{A^{3}}{2}(1-2e^{2u})\Omega^{-5}\\
&\quad + \left[A^{4}\left(\frac{5}{8} - \frac{7}{8}e^{2u} - \frac{1}{8}e^{4u}\right) - \frac{\sigma^{0}\bar{\sigma}^{0}}{2}\right]\Omega^{-6}\\
&\quad + \frac{1}{3}\Big[A^{5}\left(-\frac{37}{8} + 8e^{2u} + 8e^{2u} + 2e^{4u}\right) - \phi_{00}^{0}\\
&\quad + \bar{\sigma}^{0}(3A\sigma^{0} + \psi_{0}^{0}) + \sigma^{0}(3A\bar{\sigma}^{0} + \bar{\psi}_{0}^{0})\Big]\Omega^{-5}\\
&\quad + O(\Omega^{-8}) \qquad (7.23)\\
\sigma &= \sigma^{0}\Omega^{-4} - (2A\sigma^{0} + \psi_{0}^{0})\Omega^{-5} + O(\Omega^{-6}) \qquad (7.24)\\
\tau &= -\frac{1}{3}(\phi_{01}^{0} + \psi_{1}^{0})\Omega^{-5} + O(\Omega^{-6}\log\Omega) \qquad (7.26)
\end{aligned}
$$

$$\omega = \frac{e^{u}}{4}\left(S\bar\nabla\sigma^\circ - 2\sigma^\circ\bar\nabla S\right)\Omega^{-2} - \left[\frac{Ae^{u}}{4}\left(S\bar\nabla\sigma^\circ - 2\sigma^\circ\bar\nabla S\right) + \frac{1}{3}\left(\phi^\circ_{01} + \psi^\circ_{1}\right)\right]\Omega^{-3} + O(\Omega^{-4}\log\Omega) \qquad (7.26)$$

$$\gamma = -\frac{1}{2} - \frac{A}{2}\Omega^{-1} + \frac{A^2}{4}\Omega^{-2} - \frac{A^3}{4}\Omega^{-3} + \frac{1}{4}\left(\frac{5}{4}A^4 - \psi^\circ_2\right)\Omega^{-4} + O(\Omega^{-5}) \qquad (7.27)$$

$$\mu = \frac{A}{2}\Omega^{-1} - \frac{A^2}{4}(1+e^{2u})\Omega^{-2} + \frac{A^3}{4}(1+2e^{2u})\Omega^{-3} - \frac{1}{2}\left[A^4\left(\frac{5}{8} + \frac{7}{4}e^{2u} + \frac{1}{8}e^{4u}\right) + \frac{\sigma^\circ}{2}\left(\bar\sigma^\circ_{,1} - \bar\sigma^\circ\right) + \psi^\circ_2\right] \times \Omega^{-4} + O(\Omega^{-5}) \qquad (7.28)$$

$$\lambda = \frac{1}{2}\left(\bar\sigma^\circ_{,1} - \bar\sigma^\circ\right)\Omega^{-2} - \frac{A\bar\sigma^\circ_{,1}}{2}\Omega^{-3} + O(\Omega^{-4}) \qquad (7.29)$$

$$\nu = -\frac{1}{2}\left(\left(\bar\sigma^\circ_{,1} - \bar\sigma^\circ\right)e^{u}\nabla S - \frac{1}{2}e^{u}S\nabla\left(\bar\sigma^\circ_{,1} - \bar\sigma^\circ\right)\right)\Omega^{-2} + O(\Omega^{-3}) \qquad (7.30)$$

$$\alpha = \frac{1}{2}e^{u}\bar\nabla S\Omega^{-2} - \frac{1}{2}Ae^{u}\bar\nabla S\Omega^{-3} + \frac{1}{4}\left[A^2e^{u}\nabla S\left(\frac{5}{2} + \frac{1}{2}e^{u}\right) + e^{u}(\nabla S)\bar\sigma^\circ\right]\Omega^{-4} + O(\Omega^{-5}) \qquad (7.31)$$

$$\xi^3 = e^u S \Omega^{-2} - A e^u S \Omega^{-3} + \tfrac{1}{2}\left[A^2 e^u S\left(\tfrac{5}{2} + \tfrac{1}{2} e^u\right) - e^u S \sigma^0\right]\Omega^{-4} + O(\Omega^{-5}) \qquad (7.32)$$

$$U = \tfrac{1}{2}\Omega^2 - \tfrac{1}{8}(\psi_2^0 + \bar{\psi}_2^0)\Omega^{-2} + O(\Omega^{-3}) \qquad (7.33)$$

$$X^3 = \frac{e^u S}{15}\left[\phi_{01}^0 + \bar{\phi}_{01}^0 + \psi_1^0 + \bar{\psi}_1^0\right]\Omega^{-5} + O(\Omega^{-6}\log\Omega) \qquad (7.34)$$

$$\begin{aligned}\psi_4^0 = {} & \left[-\bar{\sigma}^0 + \tfrac{3}{2}\sigma^0_{,1} + -\tfrac{1}{2}\sigma^0_{,11}\right]\Omega^{-2} + \left[-\frac{\bar{\sigma}^0}{2} - \tfrac{3}{4}\bar{\sigma}^0_{,1} + \tfrac{1}{2}\bar{\sigma}^0_{,11}\right]\Omega^{-3} + \Big[A^2\left(\bar{\sigma}^0 + \tfrac{3}{4}\bar{\sigma}^0_{,1} - \tfrac{3}{8}\bar{\sigma}^0_{,11}\right) \\ & + \frac{A^2}{4}\left(-\bar{\sigma}^0 + \tfrac{3}{2}\bar{\sigma}^0_{,1} - \tfrac{1}{2}\bar{\sigma}^0_{,11}\right)e^{2u} - e^{2u}\bar{\nabla}\Big(S \\ & \times (\bar{\sigma}^0_{,1} - \bar{\sigma}^0)^{\frac{3}{2}}\,\nabla\big(S(\bar{\sigma}^0_{,1} - \bar{\sigma}^0)^{\frac{1}{2}}\big)\Big) \\ & + \frac{3A}{8}\bar{\psi}_0^0\Big]\Omega^{-4} + O(\Omega^{-5})\end{aligned} \qquad (7.35)$$

$$\begin{aligned}\psi^3 = {} & \left[e^u(\bar{\sigma}^0_{,1} - \bar{\sigma}^0)\nabla S - e^u S\nabla(\bar{\sigma}^0_{,1} - \bar{\sigma}^0)\right]\Omega^{-4} \\ & + \Big[Ae^u\big(-2(\bar{\sigma}^0_{,1} - \tfrac{11}{8}\bar{\sigma}^0)\nabla S - S\nabla(\bar{\sigma}^0_{,1} - \tfrac{11}{8}\bar{\sigma}^0)\big) \\ & + \tfrac{1}{2}\bar{\phi}^0_{01}\Big]\Omega^{-5} + O(\Omega^{-6})\end{aligned} \qquad (7.36)$$

$$\Psi_2 = \Psi_2^0 \Omega^{-6} + \Big[-\tfrac{3}{2} A \Psi_2^0 - e^{u} S^2 \bar{\nabla}\Big(\frac{\Psi_1^0 - \phi_{01}^0}{S}\Big)$$
$$+ \tfrac{1}{2}\Psi_0^0(\bar{\sigma}_{,1}^0 - \bar{\sigma}^0) + \frac{3A}{4}\sigma^0\bar{\sigma}_{,1}^0 - \frac{3A}{2}\sigma^0\bar{\sigma}^0$$
$$+ 16\Lambda^0\Big]\Omega^{-7} + O(\Omega^{-8}\log\Omega) \qquad \underline{(7.37)}$$

where:

$$\Psi_2^0 - \bar{\Psi}_2^0 = \Big[e^{2u}\Big(\frac{S^2}{2}\bar{\nabla}\bar{\nabla}\sigma^0 - S(\bar{\nabla}S)(\bar{\nabla}\sigma^0)$$
$$+ \sigma^0(\bar{\nabla}S)^2 - \sigma^0 S\bar{\nabla}\bar{\nabla}S\Big) - \tfrac{1}{2}\sigma^0\bar{\sigma}_{,1}^0\Big] - c.c.$$

$\Psi_2^0 + \bar{\Psi}_2^0$ is undetermined $\qquad \underline{(7.38)}$

$$\Psi_1 = \Psi_1^0\Omega^{-7} + \Psi_1^1\Omega^{-8}\log\Omega + \Psi_1^2\Omega^{-8} + O(\Omega^{-9}\log\Omega)$$

where

$$\Psi_1^0 = e^{u}\Big[S\bar{\nabla}\Big(\Psi_0^0 + \frac{15}{4}\sigma^0\Big) - \Big(2\Psi_0^0 + \frac{15}{2}\sigma^0\Big)\bar{\nabla}S\Big]$$
$$- 3\phi_{01}^0 \qquad \underline{(7.39)}$$

$$\Psi_1^1 = -e^{u}(S\bar{\nabla} - 2\bar{\nabla}S)(5A\Psi_0^0 - \Psi_0^1 + 15A\sigma^0)$$
$$+ 4\phi_{01}^1 \qquad \underline{(7.40)}$$

Ψ_1^2 is undetermined

By $\underline{(3.54)}$ $\qquad \Psi_{1,1}^1 = 3\Psi_1^1 \qquad \underline{(7.41)}$

Thus the u derivative of $\psi_1^{\,1}$, depends only on itself and not on the radiation field. It therefore represents a type of disturbance unconnected with radiation. If it is zero on one hypersurface, it will remain zero. In this case it is possible to continue the expansions of all quantities in negative powers of Ω without any log terms appearing.

8. The Asymptotic Group

The metric has the form:

$$g^{11} = g^{13} = g^{14} = 0, \quad g^{12} = 1$$
$$g^{22} = \Omega^{2} + O(\Omega^{-2})$$
$$g^{2i} = O(\Omega^{-4})$$
$$g^{ij} = -2P^{2}\delta^{ij}\Omega^{-4} + 2AP^{2}\delta^{ij}\Omega^{-5} + O(\Omega^{-6})$$

The asymptotic group is the group of coordinate transformations that leave the form of the metric and of the boundary conditions unchanged. It can be derived most simply by considering the corresponding infinitesimal transformations:

$$x'^{\mu} = x^{\mu} + \varepsilon K^{\mu}(x^{\nu}) \qquad (8.1)$$

Then $$\bar{\delta} g^{\mu\nu} = g'^{\mu\nu}(x) - g^{\mu\nu}(x)$$
$$= \epsilon\left(g^{\mu\alpha} K^{\nu}_{,\alpha} + g^{\alpha\nu} K^{\mu}_{,\alpha} - g^{\mu\nu}_{,\alpha} K^{\alpha}\right) \qquad (8.2)$$

$$\bar{\delta}\Lambda = -\varepsilon \Lambda_{,\alpha} K^{\alpha} \qquad (8.3)$$

$$\bar{\delta}\phi_{00} = \epsilon R_{1\alpha} K^{\alpha}_{,1} + \tfrac{1}{2}\epsilon R_{11,\alpha} K^{\alpha} \qquad (8.4)$$

$$\bar{\delta}\phi_{01} = \tfrac{1}{2}\epsilon\left(R_{1\alpha} K^{\alpha}_{,3} + R_{3\alpha} K^{\alpha}_{,1} + R_{13,\alpha} K^{\alpha}\right) \qquad (8.5)$$

To obtain the asymptotic group we demand

$$\begin{aligned}
\bar{\delta} g^{11} &= \bar{\delta} g^{12} = \bar{\delta} g^{13} = \bar{\delta} g^{14} = 0 \\
\bar{\delta} g^{22} &= O(\Omega^{-2}) \\
\bar{\delta} g^{2i} &= O(\Omega^{-4}) \\
\bar{\delta} g^{ij} &= O(\Omega^{-6}) \\
\bar{\delta}\Lambda &= O(\Omega^{-7}) \\
\bar{\delta}\phi_{00} &= O(\Omega^{-9}) \\
\bar{\delta}\phi_{01} &= O(\Omega^{-7})
\end{aligned} \qquad (8.6)$$

By $\bar{\delta} g^{11} = 0$,
$$K^{1} = K^{01}(u, x^{i}) \qquad (8.7)$$

By $\bar{\delta}\Lambda = O(\Omega^{-7})$

$K^2 = O(\Omega^{-2})$

$$\bar{\delta} g^{12} = 0 = K^2_{,2} + K^1_{,1} + g^{23} K^1_{,3} + g^{24} K^1_{,4} \qquad (\underline{8.8})$$

$$\therefore K^1_{,1} = 0$$

$$K^1 = K^{01}(x^i) \qquad (\underline{8.9})$$

$$\bar{\delta} g^{13} = 0 = K^3_{,2} + g^{33} K^1_{,3} + g^{34} K^1_{,4}$$

$$\therefore K^3 = K^{03}(u, x^i) - 4r^{-1} P^2 K^1_{,3} + O(\Omega^{-3}) \qquad (\underline{8.10})$$

$$\bar{\delta} g^{23} = O(\Omega^{-4}) = K^3_{,1} + 2r K^3_{,1} + O(\Omega^{-4})$$

$$\therefore K^{03} = K^{03}(x^i) \qquad (\underline{8.11})$$

$$\bar{\delta} g^{ij} = g^{i2} K^j_{,2} + g^{j2} K^i_{,2} + g^{ik} K^j_{,k} + g^{jk} K^i_{,k} - g^{ij}_{,k} K^k$$

$$\therefore K^{03}_{,4} = -K^{04}_{,3} \qquad (\underline{8.12})$$

$$\therefore K^{03}_{,3} = K^{04}_{,4} \qquad (\underline{8.13})$$

$$2S K^{03}_{,3} - S_{,3} K^{03} - S_{,4} K^{04} - 2S K^1 = 0 \qquad (\underline{8.14})$$

(8.12) and (8.13) imply that K^{0i} is an analytic function of $x^3 + i x^4$. This is a consequence if the fact that we have reduced the leading term of g^{ij} to a conformally flat form. Thus the only allowed transformations of x^i are the conformal transformations of the form:

$$x^3 + i x^4 = \frac{a(x^3 + i x^4) + b}{c(x^3 + i x^4) + d} \qquad (8.15)$$

$$ad - bc = 1$$

When the six parameters a, b, c, d are given K^1 is uniquely determined by (8.14) K^2 is also uniquely determined. Thus the asymptotic group is isomorphic to the conformal group in two dimensions. Sachs(7) has shown that this is isomorphic to the homogeneous Lorentz group. It is also however isomorphic to the group of motions of a 3-space of constant negative curvature which is the group of the unperturbed Robertson-Walker space. Thus the asymptotic group is the same as the group of the undisturbed space. It is not enlarged by the presence of radiation. This is interesting because in the case of gravitational radiation in empty, asymptotically flat space, it turns out that the asymptotic group contains not only the 10 dimensional inhomogeneous Lorentz group, the group of motions of flat space, but also infinite dimensional "supertranslations". It has been suggested that these supertranslations might have some physical significance in elementary particle physics. The above result would seem to indicate that this is probably not the case since our universe is almost certainly not asymptotically flat though it may be asymptotically Robertson-Walker.

9. What an observer would measure

The velocity vector $\underset{m}{V}$ of an observer moving with the dust will be:

$$\begin{aligned}
\underset{1}{V} &= \Omega^{-1} + O(\Omega^{-5}) \\
\underset{2}{V} &= \tfrac{1}{2}\Omega + O(\Omega^{-3}) \\
\underset{3}{V} &= O(\Omega^{-3}) \\
\underset{4}{V} &= O(\Omega^{-3})
\end{aligned} \qquad (9.1)$$

$\underset{m}{q}$, the projection of the wave vector l^a in the observer's rest-space (the apparent direction of the wave) will be:

$$\underset{m}{q} = \underset{m}{\overset{2}{\delta}} - \underset{m}{V}\,\underset{1}{V}$$

$$\therefore \underset{1}{q} = \Omega^{-2} + O(\Omega^{-5})$$

$$\underset{2}{q} = \tfrac{1}{2} + O(\Omega^{-4})$$

$$\underset{3}{q} = O(\Omega^{-4})$$
$$\underset{4}{q} = O(\Omega^{-4}) \qquad (9.2)$$

The observer's orthonormal tetrad may be completed by two space-like unit vectors $\underset{m}{S}$ and $\underset{m}{t}$

$$\underset{1}{S} = O(\Omega^{-4}) \qquad \underset{1}{t} = O(\Omega^{-4})$$
$$\underset{2}{S} = O(\Omega^{-2}) \qquad \underset{2}{t} = O(\Omega^{-2})$$
$$\underset{3}{S} = \frac{1}{\sqrt{2}} \qquad \underset{3}{t} = \frac{-i}{\sqrt{2}}$$
$$\underset{4}{S} = \frac{1}{\sqrt{2}} \qquad \underset{4}{t} = \frac{i}{\sqrt{2}} \qquad (9.3)$$

We write ${}_{a}e^{\alpha} = (V^{\alpha}, q^{a}, S^{a}, t^{\alpha})$

By measuring the relative accelerations of neighbouring dust particles, the observer may determine the 'electric' components of the gravitational wave:

$$E_{ab} = -C_{ap.bq} V^{p} V^{q}$$

In the observer's tetrad this has components

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} O(\Omega^{-4}) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} O(\Omega^{-4})$$

$$+ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} O(\Omega^{-5}) + \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} O(\Omega^{-5})$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} O(\Omega^{-6}) \qquad (9.4)$$

This should be compared to the behaviour for asymptotically flat space for which

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} O(r^{-1}) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} O(r^{-1})$$

$$+ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} O(r^{-2}) + \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} O(r^{-2}) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} O(r^{-3}) \qquad (9.5)$$

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CHAPTER 4

Singularities

If the Einstein equations without cosmological constant are satisfied, a Robertson-Walker model can 'bounce' or avoid a singularity only if the pressure is less than minus one-third the density. This is clearly not a property possessed by normal matter though it might be possessed by a field of negative energy density like the 'C' field. However there is a grave quantum-mechanical difficulty associated with the existence of negative energy density, for there would be nothing to prevent the creation, in a given volume of space-time, of an infinite number of quanta of the negative energy field and a corresponding infinity of particles of positive energy. If we therefore exclude such fields, all Robertson-Walker models must be of the 'big-bang' type, that is they have a singularity in the past and maybe one in the future as well. It has been suggested [1] that the occurrence of these singularities is a consequence of the high degree of symmetry of the Robertson-Walker models which restricts the expansion and contraction so that they are purely radial and that more realistic models with fewer or no exact symmetries would not have a singularity. This chapter will be devoted to an examination of this question and it will be shown that provided certain physically reasonable conditions hold, any model must have a singularity, that is, it cannot be a geodesically complete C^1, piecewise C^2 manifold.

2. The Fundamental Equation

The expansion $\theta = V^a{}_{;a}$ of a time-like geodesic congruence with unit tangent vector V^a obeys equation (7) of Chapter 2:

$$\theta_{;a} V^a = -\tfrac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 - R_{ab} V^a V^b \quad (1)$$

A point q will be said to be a singular point on a geodesic γ of a time-like geodesic congruence if θ for the congruence is infinite on γ at q. A point q will be said to be conjugate to a point p along a geodesic γ if it is a singular point on γ of the congruence of all time-like geodesics through p. A point q will be said to conjugate to a space-like hypersurface H^3 if it is a singular point of the congruence of geodesic normals to H^3. An alternative description of conjugate points may be given as follows: let K^a be a vector connecting points corresponding distances along two neighbouring geodesics in a congruence with unit tangent vector V^a. Then K^a is 'dragged' along by the congruence, that is

$$\mathcal{L} K^a = 0$$

$$\therefore \quad \frac{DK^a}{Ds} = V_{a;b} K^b \qquad (2)$$

$$\therefore \quad \frac{D^2 K^a}{Ds^2} = R^a{}_{bcd} V^b V^c K^d \qquad (3)$$

Introducing an orthonormal tetrad $\underset{m}{e}{}^a$ parallelly transported along V^a with $\underset{4}{e}{}^a = V^a$ we have

$$\frac{d^2 \underset{m}{K}}{ds^2} = -\underset{m}{\overset{n}{a}} \underset{n}{K} \qquad (4)$$

where $\underset{m}{\overset{n}{a}} = \overset{n}{e}{}^a \underset{m}{e}{}^b R_{acbd} V^c V^d$

A solution of (4) will be called a Jacobi field. There are clearly eight independent solutions. Since V^a and sV^a are solutions, the other six independent solutions of (4) may be taken orthogonal to V^a. Then q is conjugate to p along a geodesic γ if, and only if, there is a Jacobi field along γ which vanishes at p and q. This may be shown as follows: the Jacobi fields which vanish at p may be regarded as generating neighbouring geodesics in the irrotational congruence of all time-like geodesics through p. Therefore they obey

$$\frac{dK_m}{ds} = V_{m;n} K^n$$

They may be written

$$K_m = A_{mn}(s) \left.\frac{dK^n}{ds}\right|_p$$

where

$$\frac{dA_{mn}}{ds} = V_{m;r} A^r_n$$

Near p, $A_{mn}(s)$ will be positive definite. There will be a Jacobi field vanishing at p and q if, and only if, $\det(A_{mn}(q)) = 0$

But $A_{mn}(s) = \exp\left(\int_p^s V_{m;n}\, ds'\right)$

Therefore

$$\theta = \frac{1}{\det(A_{mn})} \frac{d}{ds} \left(\det(A_{mn})\right)$$

and

$$\frac{d^2 A_{mn}}{ds^2} = -a_{mr} A^r_n$$

Therefore $\frac{d}{ds}(A_{mn})$ is finite

Hence θ is infinite where and only where $\det(A_{mn}) = 0$.

Thus the two definitions of conjugate points are equivalent. This also shows that singular points of congruences are points where neighbouring geodesics intersect.

For null geodesic congruences with parallelly transported tangent vector l^a we may define the convergence ρ as in Chapter 3. This obeys

$$\rho_{;a} l^a = \rho^2 + \sigma\bar{\sigma} + \frac{1}{2} R_{ab} l^a l^b \quad (5)$$

We define a singular point of a null geodesic congruence as one where ρ is infinite.

The condition that the pressure is greater than minus one-third the density may be stated more generally as condition (a).

(a) $E > 0$, $E > \frac{1}{2} T$, for any observer with 4-velocity w^a, where $E = T_{ab} w^a w^b$ is the energy density in the rest-frame of the observer and $T = T^a{}_a$ is the rest-mass density.

Condition (a) will be satisfied by a perfect fluid with density $\mu > 0$ and pressure $p > -\frac{1}{3}\mu$. It implies $R_{ab} V^a V^b \geq 0$ for any time-like or null vector V^a. Therefore by equations (1) and (5) any time-like or null irrotational geodesic congruence must have a singular point on each geodesic within a finite affine distance. Obviously if the flow-lines form an irrotational geodesic congruence, there will be a physical singularity at the singular points of the congruence where the density and hence the curvature are infinite. This will be the case if the universe is filled with non-rotating dust [2,3]. However, if the flow-lines are not geodesic (ie. non-vanishing pressure gradient) or are rotating, equation (1) cannot be applied directly.

3. Spatially Homogeneous Anisotropic Universes

The Robertson-Walker models are spatially homogeneous and isotropic, that is, they have a six parameter group of motions transitive on a space-like surface. If we reduce the symmetry by considering models that are spatially homogeneous but anisotropic (that is, they have a three parameter group of motions transitive on a space-like hypersurface) then the matter flow may have rotation, acceleration and shear. Thus there would seem to be the possibility of non-singular models. L. Shepley[4] has investigated one particular homogeneous model containing rotating dust and has shown that there is always a singularity. Here a general result will be proved.

There must be a singularity in every model which satisfies condition (a) and,

(b) there exists a G_r of motions on the space or on universal covering space *, $r \geq 3$ which is transitive on at

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* See section 5

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least one space-like surface but space-time is not stationary,

(c) the energy-momentum tensor is that of a perfect fluid, $T_{ab} = (\mu + p) u_a u_b - p g_{ab}$. u^a is the tangent to flow-lines and is uniquely defined as the time-like eigen-vector of the Ricci tensor.

PROOF

R, the curvature scalar must be constant on a space-like surface of transitivity H^3 of the group. Therefore $R_{;a}$ must be in the direction of the unit time-like normal V^a to H^3 .

$$V_a = \frac{e(R_{;a})\, R_{;a}}{\sqrt{f}}$$

where $R_{;a} R^{;a} = f > 0$

$e(R_{;a})$ is an indicator $= +1$ if $R_{;a}$ is past directed
$= -1$ if $R_{;a}$ is future directed

Then $V_{[a;b]} = 0$.

Thus V_a is a congruence of geodesic irrotational time-like vectors. By condition (a), $R_{ab} V^a V^b > 0$.

Therefore the congruence must have a singular point on each geodesic (by equation 1) either in the future or in the past. Further, by the homogeneity, the distance along each geodesic from H^3 to the singular point must be the same for each geodesic. Thus if the surfaces of transitivity remain space-like, they must degenerate into, at the most, a 2-surface C^2 which will be uniquely defined . Let M be the subset of the flow-lines of the matter which intersect C^2 . Let L be the non-empty subset of H^3 intersected by M . Since there is a group transitive on H , L must be H^3 itself. Thus <u>all</u> the flow-lines through H^3 must intersect the 2-surface C^2. Thus the density will be infinite there and there will be a <u>physical</u> singularity. Alternatively if the surfaces of transitivity do not remain space-like, there must be at least one surface which is null - call this S^3. At S^3, $\not{F} = 0$, $R_{;a} \neq 0$ (if $R_{;a}$ is zero, we can take any other scalar polynomial in the curvature tensor and its covariant derivatives. They cannot all be zero if space-time is not stationary). We introduce a geodesic irrotational null congruence on S^3 with tangent vector L^a where $L_a \; R_{;a}$. Then by equation (5), there will be a singular point of each null geodesic in S^3 within finite affine distance either in the future or in the past. The 2-surface of these singular points will be uniquely defined. The same argument used before shows that the density becomes infinite and there is a physical singularity. In fact as S^3 is a surface of homogeneity, the whole of S^3 will be singular and it is not meaningful to call it null or to distinguish this case from the case where the surfaces of transitivity remain space-like.

The conditions (a), (b), (c) may be weakened in two ways. Condition (b) that there is a group of motions throughout space-time may be replaced by (b′) and (d).

(b') There is a space-like hypersurface H^3 in which there are three independent vector fields $\underset{m}{X}^a$ such that

$\pounds_{\underset{m}{X}^a} g_{bc} = 0, \ \pounds_{\underset{m}{X}^a} R_{bcde} = 0$ on H^3. That is, there is one homogeneous space section.

(d) There exist equations of state such that the Cauchy development of H^3 is determinate.

Then succeeding space-like surfaces of constant R are homogeneous and much the same proof can be given that there are no non-singular models satisfying (a), (b'), (c), (d).

The only property of perfect fluids that has been used in the above proof is that they have well defined flow-lines intersection of which implies a physical singularity. Obviously, however, this property will be possessed by a much more general class of fluids. For these, we define the flow vector as the time-like eigenvector (assumed unique) of the energy-momentum tensor. Then we can replace condition (c) on the nature of the matter by the much weaker condition (e).

(e) If the model is singularity-free, the flow-lines form a smooth time-like congruence with no singular points with a line through each point of space-time.

Condition (e) will be satisfied automatically if conditions (a) and (c) are.

This proof rests strongly on the assumption of homogeneity which is clearly not satisfied by the physical universe locally though it may hold on a large enough scale. However it would seem to indicate that large scale effects like rotation cannot prevent the singularity.

It is of interest to examine the nature of the singularity in the homogeneous anisotropic models since this is more likely to be representative of the general case than that of the isotropic models. It seems that in general the collapse will be in one direction,[5] that is, the universe will collapse down to a 2-surface. Near the singularity, the volume will be proportional to the time from the singularity irrespective of the precise nature of the matter. It also appears that the nature of the particle horizon is different. There will be a particle horizon in every direction except that in which the collapse is taking place.

4. Singularities in Inhomogeneous Models

Lifshitz and Khalatnikov[6] claim to have proved that a general solution of the field equations will not have a singularity. Their method is to contract a solution with a singularity which they claim is representative of the general solution with a singularity, and then show that it has one fewer arbitrary function than a fully general solution.

Clearly their whole proof rests on whether their solution is fully representative and of that they give no proof. Indeed it would seem that it is not representative since it involves collapse in two directions to a 1-surface whereas in general one would expect collapse in one direction to a 2-surface. In fact their claim has been proved false by Penrose[7] for the case of a collapsing star using the notion of a 'closed trapped surface'. A similar method will be used to prove the occurrence of singularities in 'open ' universe models.

5. 'Open' and 'Closed' Models

The method used by Penrose to prove the occurrence of a physical singularity depends on the existence of a non-compact Cauchy surface. A Cauchy surface will be taken to mean a complete, connected space-like surface that intersects every time-like and null line once and once only. Not all spaces possess a Cauchy surface: examples of those that do not include the plane-wave metrics,[8] the Godel model,[9] and N.U.T. space.[10] However none of these have any physical significance. Indeed it would seem reasonable to demand of any physically realistic model that it possess a Cauchy surface. If the Cauchy surface is compact, the model is commonly said to be 'closed'; if non-compact, it is said to to be 'open'. The surfaces, t = constant, in the Robertson-Walker solutions for normal matter are examples of Cauchy surfaces. If $K = -1$, they have negative curvature and it is frequently stated that they are non-compact. This is not necessarily so: there exist possible topologies for which they are compact. However, the following statements may be made about the topology of the surfaces t = constant.

If the curvature is negative, $K = -1$, the universal covering space is non-compact and is diffeomorphic to E^3.[11] Any other topology can be obtained by identification of points. Thus any other topology will not be simply connected and, if compact, must have elements of infinite order in the fundamental group. Further if compact, they can have no group of motions.[12]

If the curvature is zero, $K = 0$, the universal covering space is E^3. There are eighteen possible topologies.[13] If compact they have a G_3 of motions and Betti numbers, $B_1 = 3$, $B_2 = 3$.[12]

If the curvature is positive, $K = +1$, the universal covering space is S^3. Thus all topologies are compact. The Betti numbers are all zero.[12]

Since a singularity in the universal covering space implies a singularity in the space covered, Penrose's method is applicable not only to spaces that have a non-compact Cauchy surface but also to spaces whose universal covering space has a non-compact Cauchy surface. Thus it is applicable to models which, at the present time, are homogeneous and isotropic on a large scale with surfaces of approximate homogeneity which have negative or zero curvature.

6. The Closed Trapped Surface

Let T^3 be a 3-ball of coordinate radius r in a 3-surface H^3 (t = const.) in a Robertson-Walker metric with K = 0 or -1. Let q^a be the outward directed unit normal to T^2, the boundary of T^3, in H^3 and let V^a be the past directed unit normal to H^3. Consider the outgoing family of null geodesics which intersect T^2 orthogonally. At T^2, ρ, their convergence will be:

$$\rho = \tfrac{1}{2}(V_{a;b} + q_{a;b})(S^a S^b + t^a t^b)$$

where S^a t^a are unit space-like vectors in H^3 orthogonal to q^a and to each other,

therefore $$\rho = \frac{2}{R}\left[\sqrt{\frac{\mu}{3} - K} - \frac{1}{r}\sqrt{1 - Kr^2}\right]$$

If $\mu > 0$ and K = 0 or -1, by taking r large enough, we may make $-\rho$ negative at T^2. Therefore, in the language of Penrose, T^2 is a closed trapped surface.

Another way of seeing this is to consider the diagram in which the flow-lines are drawn at their proper spatial distance from an observer. They all meet in the singularity at $t = 0$. If the past light cone of the observer is drawn on this diagram, it initially diverges from his world-line ($\rho < 0$). It reaches a maximum proper radius ($\rho = 0$) and then converges again to the singularity ($\rho > 0$). The intersection of the converging light cone and the surface H^3 gives a closed trapped surface T^2. If the red-shift of the quasi-stellar 3C9 is cosmological then it will be beyond the point $\rho = 0$ if we are living in a Robertson-Walker type universe with normal matter. However, the assumptions of homogeneity and isotropy in the large seem to hold out to the distance of 3C9. Thus there is good reason to believe that our universe does in fact contain a closed trapped surface. It should be pointed out that the possession of a closed trapped surface is a large scale property that does not depend on the exact local metric. Thus a model that had local irregularities, rotation and shear but was similar on a large scale at the present time to a Robertson-Walker model would have a closed trapped surface.

Following Penrose it will be shown that space-time has a singularity if there is a closed trapped surface and :

(f) $E \geqslant 0$ for any observer with velocity

(g) there is a global time orientation

(h) the universal covering space has a non-compact Cauchy surface H^3.

PROOF

Assume space-time is singularity free. Let F^4 be the set of points to the past of H^3 that can be joined by a smooth future directed time-like line to T^2 or its interior T^3. Let B^3 be the boundary of F^4. Local considerations show that $B^3 - T^3$ is null where it is non-singular and is generated by the outgoing family of past directed null geodesic segments which have future end-point on T^2 and past end-point where or before a singular point of the null geodesic congruence. Since at T^2, the convergence, $\rho > 0$ and since $R_{ab} L^a L^b \geq 0$ by (f), the convergence must become infinite within finite affine distance. Thus $B^3 - T^3$ will be compact being generated by a compact family of compact segments. Hence B^3 will be compact. Penrose's method is then as follows: approximate B^3 arbitrarily closely by a smooth space-like surface and project B^3 onto H^3 by the normals to this surface. This gives a many-one continuous mapping of B^3 into H^3. Since B^3 is compact, its image B^{3*} must be compact. Let $d(Q)$ be the number of points of B^3 mapped to a point Q of H^3. $d(Q)$ will change only at the intersection of caustics of the normals with H^3. Moreover, by continuity $d(Q)$ can only change by an even number.

But $d(T^3)$ = 1 since this is the identity and $d(H^3 - B^{3*}) = 0$. This is a contradiction, thus the assumption that space-time is non-singular must be false. An alternative proceedure which avoids the slightly questionable step of approximating B^3 by a space-like surface is possible if we adopt condition (e) on the nature of the matter, then B^3 may be projected continuously one-to-one onto H^3 by the flow-lines. This again leads to a contradiction since B^3 is compact and H^3 is not.

~~7. Cauchy Horizons~~

~~In the above proof it was necessary to demand that H^3 be a Cauchy surface otherwise the whole of B^3 might not have been projected onto H^3. We will define a semi-Cauchy surface (s.C.s) as a complete connected space-like surface that intersects every time-like and null line at most once. A s.C.s H^3 will be a Cauchy surface for points near it, that is, it will intersect every time-like and null line through these points. However, further away there may be regions for which it is not a Cauchy surface. Let P^4 be the set of points for which H^3 is a Cauchy surface and let Q^3 be the boundary of these points. Q^3, if it exists, will be called the Cauchy Horizon relative to H^3. Q^3 must be a null surface. Furthermore if condition (f) holds the null geodesics~~

~~generating Q^3 must have at least one singular point which means that they must be bounded in at least one direction.~~ ~~A simple example of s.C.s with a Cauchy horizon is a space-like surface of constant negative curvature completely within the null-cone of the point in Minkowski space. The null-cone forms the Cauchy horizon.~~

~~If conditions (e) and (g) hold, then a model with a compact s.C.s, H^3 must have the topology: $H^3 \times E^1$.~~

~~PROOF~~

~~Suppose there were a region V^4 through which there were no flow-lines intersecting H^3, then V^3 the boundary of V^4 must be a time-like surface generated by flow-lines which do intersect H^3. Proceeding along these flow-lines in the direction of their intersection with H^3, we must reach an end-point of the generator since V^3 does not intersect H^3. But the existence of this end-point contradicts (e) since it implies a singularity of the flow-line congruence. Thus V^4 must be empty and every point has a flow-line through it intersecting H^3. Thus we have a homeomorphism of the space to $H^3 \times E^1$ by assigning to every point of the space its distance along the flow-line from H^3 and the point of intersection of the flow-line with H^3. It can also be shown that~~

~~in this case H^3 must be a Cauchy surface. For, suppose there were a Cauchy horizon $\mathcal{C}^3$; this can intersect each flow-line at most once. Therefore there is a homeomorphism of $\mathcal{C}^3$ into H^3. Further, by (e) every flow-line must intersect $\mathcal{C}^3$. Thus $\mathcal{C}^3$ is homeomorphic to H^3 and is compact. If condition (f) holds every null geodesic generator of $\mathcal{C}^3$ has at least one end-point. This must be in the direction away from H^3 since in the direction towards H^3 each generator must be unbounded. This however is impossible since $\mathcal{C}^3$ is compact. Thus H^3 is a Cauchy surface.~~

8. Singularities in 'Closed' Universes

There is a singularity in every model which satisfies (a), (g) and (i).

(i) There exists a compact Cauchy surface H^3 whose unit normal V^a has positive expansion everywhere on H^3.

PROOF

For the proof it is necessary to establish a couple of lemmas. Assume that space-time is singularity-free. The following result is quoted without proof, it may ~~readily~~ be derived from lemmas proved in reference 11.

If p and q are conjugate points along a geodesic γ and x is a point on γ not in pq then x must have a conjugate point in pq.

An immediate corrollary is that if q is the first point along γ conjugate to p and y is in pq then y has no conjugate points in pq . Also since the result that x has a conjugate point in pq can only depend on the values of q_{mn} in pq, any irrotational geodesic congruence including the geodesic γ must have a singular point on γ in pq. Thus if q is a point on M^3 and γ is the geodesic normal to M^3 through q, then a point conjugate to q along γ cannot occur until after a point conjugate to M^3.

If M^3 is a complete connected space-like surface which intersects every time-like and null line from a point p, we may define a function over M^3 as the square of the geodesic distance from p which is taken as positive if the geodesic is time-like and negative if the geodesic is space-like. We call this the world function σ with respect to p. For the closed set of values $\sigma \geqslant 0$, σ will be a continuous (in general multi-valued) function over M^3. A time-like geodesic γ from p will be said to be <u>critical</u> if it corresponds to a value of σ for which $\sigma_{;\mu} \, {}_{i}\xi^{\mu} = 0 \; (i = 1, 2, 3)$ where ${}_{i}\xi^{\mu}$ are three independent vectors in M^3. Clearly a critical geodesic must be orthogonal to M^3. A geodesic which is critical will be said to be maximal if it corresponds to a local maximum of   .

Lemma 1.

A geodesic γ cannot be maximal for a smooth M^3 if there is a point x conjugate to M^3 but no point conjugate to q on γ in qp, where q is the intersection of γ and M^3.

Let f_m and g_m be the Jacobi fields along γ which vanish at x and p respectively. They may be written

$$f_m = A_{mn}(s)\, f^n/q\ ,$$

$$g_m = B_{mn}(s)\, g^n/q\ .$$

Then $h^m \left(\frac{dA_{mn}}{ds}\Big|_q - \frac{dB_{mn}}{ds}\Big|_q \right) h^n$ must be positive for any h_n since if it were negative for any h_n by taking $a_{mb} = -b\, h_m h_n$ beyond q, it would be possible to have a point y on γ beyond q conjugate to x before a point conjugate to p . If it were zero x would be conjugate to p . This shows that the surface at q of constant geodesic distance from p lies nearer to p in every direction than the surface of q of constant geodesic distance from x does. Since x is conjugate to M^3 the surface at q of constant geodesic distance from p lies closer to $\cancel{M^3}$ p in some direction than M^3 does. Hence γ is not maximal.

~~must be positive definite at x. But at x~~

$$\left(A_{mn} - B_{mn}\right)\delta^{m}\delta^{n} = B_{mn}\delta^{m}\delta^{n} < 0$$

~~Thus~~ $\frac{d}{ds}\left(A_{mn} - B_{mn}\right)$ ~~cannot be positive definite at q. Therefore there must be some direction~~ K^{m} ~~for which~~

$$K^{m}K^{n}\frac{d}{ds}A_{mn} < K^{m}K^{n}\frac{d}{ds}B_{mn}$$

~~and~~ $$K^{m}K^{n}V_{m;n} < K^{m}K^{n}W_{m;n}$$

~~at q, where~~ W_{m} ~~is the unit tangent vector of the congruence of geodesics through p. Thus in the direction~~ K ~~the surface of constant geodesic distance from p lies closer to p than the surface~~ M^3 ~~does. Therefore~~ γ ~~is not maximal~~.

If M^3 is compact or if the intersection of all time-like and null lines with M^3 is compact, σ must have a maximum value, thus there must be a geodesic normal to M^3 through p longer than γ. We use this to prove another lemma.

Lemma 2

If p lies to the future (past) on a time-like geodesic γ through q, beyond a point z conjugate to q, and there exists a compact Cauchy surface H^3 through q, then there must be another time-like geodesic from p to q longer than γ.

Let y be the last point conjugate to q on γ before p. Let x be the nearest point to p conjugate to p in pq. Let r be a point in yx. Let K^3 be the set of points which have a future (past) directed geodesic of length rq from q. Then

K^3 will be a space-like hypersurface through r. Let F^4 be the set of points which have at least one future (past) directed geodesic from q of length greater than rq. Then the boundary of F^4, $J^3 \subset K^3$. Since p is in F^4 and since every past (future) directed time-like and null line from p intersects H^3 which is not in F^4, they must also intersect J^3. Let L^3 be the intersection of J^3 and these lines. Since H^3 is compact, L^3 must be compact. Consider the function σ with respect to p over K^3. Its maximum must lie in the compact region L^3. But, by the previous lemma γ is not maximal, moreover, local considerations show that a singular point in the surface J^3 cannot be a maximum of σ. Thus the maximum value of σ must occur for a geodesic from p orthogonal to L^3. This must also be a goedesic from p to q of length greater than γ.

Using these two lemmas the theorem may be proved. Since the future (past) directed normals to H^3 are converging everywhere on H^3, there must be a point conjugate to H^3 a finite distance along each future (past) directed geodesic normal. Let β be the maximum of these distances. Let p be a point on a future (past) directed geodesic normal at a distance greater than β. Consider the function σ with respect to p over the compact surface H^3. Let λ be the geodesic from p normal to H^3 at the point q, where σ has its maximum. There must be a point conjugate to H^3 along λ in qp. But if there is no point conjugate to q along λ in qp, then λ cannot be maximal by the first lemma. If however there is a point conjugate to q along λ in qp, then there must be a longer geodesic from q to p by the second lemma. Thus is not the geodesic of maximum length from H^3 to p. This is a contradiction which shows that the original assumption that the space was non-singular must be false.

This proof could also be used to show the occurrence of a singularity in a model with a non-compact Cauchy surface provided that the expansion of its normals was bounded away from zero and provided that the intersection of the Cauchy surface with all the time-like and null lines from a point was compact.

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